Financial markets with small transaction costs

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UNSW-Macquarie University Workshop Risk: Modemling, Optimization, and Inference

Sydney, December 7, 2017

Approximate hedging

- Continuous time setting, gBm price process of a risky security.
- Classical call-option $(S_1 - K)^+$.
- Transaction costs coefficient $k_n = n^{-1/2} k_0 \to 0$.
- Leland prescription : use the BS-formula with enlarge volatility

$$\widehat{\sigma}^2 = \sigma^2 + \sigma k_0 \sqrt{8/\pi}.$$

- Equidistant grid with $H^n_{t_i} = \widehat{C}_x(t_n, S_{t_n}) = C_x(t_n, S_{t_n}, \widehat{\sigma})$

$$V^n_t = \widehat{C}(0, S_0, \widehat{\sigma}) + \int_0^t \sum_{i=1}^n H^n_{t_{i-1}} I_{]t_{i-1}, t_i]}(u) dS_u - \sum_{0<t_i<t} k_n S_{t_i} |H^n_{t_i} - H^n_{t_{i-1}}|.$$

Theorem (Leland–Lott)

$V^n_1 \to V_1 = (S_1 - K)^+$ *in probability.*

The same result, if $k_n = k_0 n^{-\alpha}$, $\alpha \in]0, 1/2]$ and the correction

$\widehat{\sigma}^2_n := \sigma^2 \left(1 + \frac{\gamma_n}{\sigma}\right)$, $\qquad \gamma_n := \sqrt{\frac{8}{\pi}} k_n n^{1/2} = \sqrt{\frac{8}{\pi}} k_0 n^{1/2-\alpha}$.

Approximation error

Theorem

The mean square approximation error is of order n^{-1}. More precisely,

$$E(V_1^n - V_1)^2 = A_1 n^{-1} + o(n^{-1}), \qquad n \to \infty,$$

where the coefficient

$$A_1 = \int_0^1 \left[\frac{\sigma^4}{2} + \sigma^3 k_0 \sqrt{\frac{2}{\pi}} + k_0^2 \sigma^2 \left(1 - \frac{2}{\pi}\right) \right] \Lambda_t dt$$

with $\Lambda_t = ES_t^4 \widehat{C}_{xx}^2(t, S_t)$. Explicitly,

$$\Lambda_t = \frac{K^2}{2\pi\widehat{\sigma}\sqrt{1-t}\sqrt{2\sigma^2 t + \widehat{\sigma}^2(1-t)}} \exp\left\{ -\frac{\left(\ln\frac{S_0}{K} - \frac{1}{2}\sigma^2 t - \frac{1}{2}\widehat{\sigma}^2(1-t)\right)^2}{2\sigma^2 t + \widehat{\sigma}^2(1-t)} \right\}.$$

Ramifications, 1
Non-uniform grid

Let f be a strictly increasing differentiable function on $[0,1]$ with $f(0)=0$, $f(1)=1$ and let $g:=f^{-1}$ denote its inverse.
The revision dates are $t_i=t_i^n=g(i/n)$, where $g:=f^{-1}$.
The enlarged volatility depends on t :

$$\widehat{\sigma}_t^2=\sigma^2+\sigma k_0\sqrt{8/\pi}\sqrt{f'(t)}.$$

The pricing function

$$\widehat{C}(t,x)=E(xe^{\xi\rho_t-\frac{1}{2}\rho_t^2}-K)^+, \qquad t\in[0,1],\ x>0,\ \xi\sim\mathcal{N}(0,1),$$

where $\rho_t^2=\int_t^1\widehat{\sigma}_s^2ds$, admits the explicit expression

$$\widehat{C}(t,x)=x\Phi(\rho_t^{-1}\ln(x/K)+\rho_t/2)-K\Phi(\rho_t^{-1}\ln(x/K)-\rho_t/2), \qquad t<1.$$

Assumption 1 : $g,f\in C^2([0,1])$.
Assumption 2 : $g(t)=1-(1-t)^\beta$, $\beta\geq 1$.

Ramifications, 2

Approximation error for non-uniform grids

Theorem

Under any of the above assumptions

$$E(V_1^n - V_1)^2 = A_1(f)n^{-1} + o(n^{-1}), \qquad n \to \infty,$$

where

$$A_1(f) = \int_0^1 \left[\frac{\sigma^4}{2} \frac{1}{f'(t)} + k_0 \sigma^3 \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{f'(t)}} + k_0^2 \sigma^2 \left(1 - \frac{2}{\pi}\right) \right] \Lambda_t dt,$$

with $\Lambda_t = ES_t^4 \widehat{C}_{xx}^2(t, S_t)$. *Explicitly,*

$$\Lambda_t = \frac{1}{2\pi\rho_t} \frac{K^2}{\sqrt{2\sigma^2 t + \rho_t^2}} \exp\left\{ - \frac{\left(\ln \frac{S_0}{K} - \frac{1}{2}\sigma^2 t - \frac{1}{2}\rho_t^2\right)^2}{2\sigma^2 t + \rho_t^2} \right\}.$$

Ramifications, 3

Convex pay-offs, non-uniform grids

Let the pay-off be $G(S_1)$. The pricing function

$$\widehat{C}(t,x) = EG(xe^{\xi\rho_t - \frac{1}{2}\rho_t^2}), \qquad t \in [0,1],\ x > 0,\ \xi \sim \mathcal{N}(0,1),$$

solves the Cauchy (terminal value) problem in the domain $[0,1]\times]0,\infty[$

$$\widehat{C}_t(t,x) + \frac{1}{2}\widehat{\sigma}_t^2 x^2 \widehat{C}_{xx}(t,x) = 0, \qquad \widehat{C}(1,x) = G(x).$$

Assumption 3 : $G : \mathbf{R}_+ \to \mathbf{R}$ is a convex function such that $G|_{I_j} \in C^2(I_j)$, where the intervals $I_j := [K_{j-1}, K_j]$, $j \le N$, $I_{N+1} := [K_N, \infty[$ and $G''(x) \le \kappa(1 + x^m)$ for some $\kappa, m > 0$.

For contingent claim satisfying Assumption 3 the previous theorem remains valid.

Ramifications, 4

Functional limit theorem

Let $\widehat{V} = \widehat{C}(t, S_t)$.

Theorem

Suppose that Assumptions 1 or 2 hold. Suppose also that Assumption 3 on the pay-off function is fulfilled. Then the distributions of the process $Y^n := n^{1/2}(V^n - \widehat{V})$ *in the Skorohod space* $\mathcal{D}[0,1]$ *converge weakly to the distribution of the process*

$$Y_t = \int_0^t F(t, S_t) dW'_t$$

where W' *is an independent Wiener process and*

$$F(t,x) = \left[\frac{\sigma^4}{2}\frac{1}{f'(t)} + k_0\sqrt{\frac{2}{\pi}}\frac{\sigma^3}{\sqrt{f'(t)}} + k_0^2\sigma^2\left(1 - \frac{2}{\pi}\right)\right]^{1/2} \widehat{C}_{xx}(t,x)x^2.$$

Geometric formulation of models transaction costs, 1

- Piece-wise constant price process $S^n = (S^n_t)_{t\in[0,1]}$ with jumps at $t_k = k/n$, $k = 1, \dots, n$, $S_0 = \mathbf{1}$, $S^0 \equiv 1$.
- Solvency cone in terms of the numéraire $K^n \in \mathbb{R}^d$ with $K^{n*} = \mathbb{R}_+(\mathbf{1} + n^{-1/2}U_n)$ where the sequence of convex compact sets U_n lays in a bounded subset of the linear subspace $\{x \in \mathbb{R}^d : \ x^0 = 0\}$ (identified with $\mathbb{R}^{d-1}$) and converges to a convex compact U. E.g., $U = [-\kappa_0, \kappa_0]$.
- Solvency cone in terms of physical units $K^n_t := K^n/S^n_t$ with the dual $\hat{K}^n_t := D^n_t K^n$, where $D^n_t := \mathrm{diag}\, S^n_t K^n$
- Price dynamics

$$\widehat{V}_t = x + \sum_{t_k \le t} \widehat{\Delta B}_{t_k}, \qquad \widehat{\Delta B}_{t_k} \in L^0(-\widehat{K}^n_{t_k}, \mathcal{F}^n_{t_k}).$$

We denote $\widehat{\mathcal{A}}^n_x(1)$ the set of terminal values of such processes.
- Consistent price systems are martingales $Z \in \mathcal{M}_0(\widehat{K}^{n*} \setminus \{0\})$, i.e $Z_t \in L^1(\widehat{K}^{*n}_t \setminus \{0\})$, Z is constant on $[t_{k-1}, t_k[$, $Z^0_0 = 1$.

Geometric formulation of models transaction costs, 2

- Contingent claim is an arbitrary $\mathbb{R}^d$-valued random variable ζ^n (a function of the price, i.e. $\zeta^n = \widehat{F}(S^n)$).
- Hedging set is the set of initial values allowing portfolio processes with the terminal values dominating the contingent claim $\widehat{F}(S^n)$, i.e.

$$\Gamma^n = \{x \in \mathbb{R}^d : \ \widehat{F}(S^n) \in \widehat{\mathcal{A}}^n_x(1)\}.$$

- The hedging theorem gives the following description :

$$\Gamma^n = \{x \in \mathbb{R}^d : \ Z_0 x \geq E Z_1 \widehat{F}(S^n) \ \ \forall Z \in \mathcal{M}(\widehat{K}^{n*} \setminus \{0\})\}.$$

- **Problem** : given a model for S^n find the limit Γ^∞ of the sets Γ^n.
- Closed limit convergence $\Gamma^n \to \Gamma^\infty$:
  1. For any $x \in \Gamma^\infty$, there are $x^n \in \Gamma^n$ such that $x^n \to x$.
  2. For any convergent subsequence of a sequence of vectors $x^n \in \Gamma^n$, its limit x belongs to Γ^∞.

Grépat theorem (simplified form)

- $d = 2$. The price process $S_t^n := (1, \eta_{[nt]})$ where
$$\eta_k^n = \prod_{j \le k} \left(1 + n^{-1/2} \sigma \xi_j^n\right),$$
ξ_k^n, $k = 1, \dots, n$, are independent and take values -1 and 1 with equal probabilities.
- $U_n = (1 + n^{-1/2}\lambda)^{-1} - 1, n^{-1/2}\lambda)$.
- $\widehat{F}$ is bounded function on the Skorohod space, Lipschitz in the uniform norm.
- Let W be a Wiener process and let g be a predictable process,
$$\sigma(\sigma - 2\lambda)^+ \le g^2 \le \sigma(\sigma + 2\lambda).$$
Put $\Gamma^\infty := \{x \in \mathbb{R}^2 \colon\; Z_0 x \ge E Z_1 \widehat{F}(Z) \;\; \forall Z = (1, \mathcal{E}(g \cdot W))\}$.

Theorem (Grépat, 2013)

$\Gamma^n \to \Gamma^\infty$.

Bank–Dolinsky–Perkkiö theorem, 1

- $d \geq 2$. Let $\mathcal{V} := \{v_1, ..., v_d\}$ be the set of vertices of a regular simplex $\Sigma \subset \mathbb{R}^{d-1}$, i.e. d affine independent vectors such that $|v_i|^2 = d-1$, $\sum_{i \leq d} v_i = 0$, and $v_i v_j = -1$ for $i \neq j$.
- The price process $S^n_t := (1, \eta_{[nt]})$ where

$$\ln \eta^n_k = \sum_{j \leq k} \ln \Big(\mathbf{1} + n^{-1/2} \sigma \xi^n_j \Big),$$

  ξ^n_k, $k = 1, \dots, n$, are independent and take values v_i with equal probabilities.
- $U_n := [-\kappa_-, \kappa_+] \subset \mathbb{R}^{d-1}$. All transaction are "buy stock" or "sell stock". The payoff function $\bar{F}(\tilde{S}^n) := \sum_i F^i(\tilde{S}^n)$ where $\tilde{S}^n$ is the process obtained from S^n by linear interpolation. is continuous and of polynomial growth in $C[0,1]$.
- The hedging price $\bar{x}^n$ is the infimum of the initial capitals $x \in \mathbb{R}$ admitting a portfolio which net worth at $T = 1$ dominates $\bar{F}(\tilde{S}^n)$.

Bank–Dolinsky–Perkkiö theorem, 2

- Let $\mathbf{G}$ be the set of $(d-1)\times(d-1)$ matrices of the matrix
$$\sigma\sigma' + \sigma\beta + \beta'\sigma'$$
where β is a matrix which column $\beta_k = \sum_{j\le d} w_{jk} v_j$ for some $w_{jk} \in [0, (1/d)(\kappa_- + \kappa_+)]$ (the set of such β is denoted by $\mathbf{B}$).
- Let W be $(d-1)$-valued Wiener process defined on some stochastic basis and let $\mathcal{H}$ be set of predictable $(d-1)$-valued processes such that $HH' \in \mathbf{G}$.

Theorem

Suppose that for any $\beta \in \mathbf{B}$ the matrix $\sigma' + \beta$ is invertible and $v_i'\beta(\sigma'+\beta)^{-1}v_j > -1$ for all i,j. Then
$$\lim \bar{x}^n = \sup_{H\in\mathcal{H}} E\bar{F}(1, \mathcal{E}(H^2\cdot W^2), \dots, \mathcal{E}(H^d\cdot W^d)).$$

Multidimensional version of the Grépat theorem

- Let $\tilde{\mathbf{G}}$ be the set of $(d-1) \times (d-1)$ positive definite matrices of the form

$$\sigma\sigma' + \sigma\beta + \beta'\sigma'$$

where $\beta = \frac{1}{d}\sum_{j \le d} v_j \otimes u_j$ (the set of such matrices β is denoted by $\tilde{\mathbf{B}}$).

- Let W be $(d-1)$-valued Wiener process defined on some stochastic basis and let $\mathcal{H}$ be the set of predictable $(d-1)$-valued processes such that $HH' \in \tilde{\mathbf{G}}$.

Theorem

Suppose that for any $\beta \in \tilde{\mathbf{B}}$ the matrix $\sigma' + \beta$ is invertible and $(v_i, \beta(\sigma'+\beta)^{-1}v_j) > -1$ for all i, j. Then $\Gamma^n \to \Gamma^\infty$ where $\Gamma^\infty = \{x \in \mathbb{R}^d : \ Z_0 x \ge EZ_1\widehat{F}(Z) \ \forall Z = (1, \mathcal{E}(H^2 \cdot W^2), ..., \mathcal{E}(H^d \cdot W^d))\}$.

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