Dynamic Risk Measures and Nonlinear Expectations with Markov Chain noise

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Outline

Risk and Expectation

- With recent events it has become increasingly clear that we need to have a good understanding of risk.
- One aspect of this is the development of simple ways of representing risk numerically.
- Some of these ways are well known – value at risk, expected shortfall, etc...
- Unfortunately, these methods are static, and fail to give consistent answers when considered at multiple time points.

- Various progress has been made in developing *dynamic risk measures*, which give time consistent answers.
- These are closely related to the g-expectations considered by Peng and others.
- Central to the mathematical study of these is the theory of Backward Stochastic Differential Equations (BSDEs).
- Most work in this area uses only noise from a Brownian motion; we here consider using a Markov chain instead.

What is a BSDE?

- Replicating portfolios are a common theme in Mathematical Finance – the classic example in a complete market is replicating the payoff of an option using a combination of the stock and a risk free asset.
- Backward Stochastic Differential Equations can be thought of as a generalisation of this.

$$Q = Y_t - \int_{]t,T]} F(\omega, u, Y_{u-}, Z_u)du + \int_{]t,T]} Z_u dM_u$$

- Q is the random ‘terminal value’
- Y_t corresponds to the value of a replicating portfolio
- Z_t corresponds to the hedging components of the portfolio
- F is a ‘driver’ function which gives the *cost* of holding the portfolio at time t, in some sense. This function can be random and non-linear.
- For simplicity, we shall assume Q, Y_t, F all take values in $\mathbb{R}$.

$$Q = Y_t - \int_{]t,T]} F(\omega, u, Y_{u-}, Z_u)du + \int_{]t,T]} Z_u dM_u$$

- M is a martingale (in $\mathbb{R}^N$), and so the last term is a *stochastic* integral.
- Z is takes values in $\mathbb{R}^{1\times N}$, so $\int Z_u dM_u$ is scalar.
- Most people consider the case where M is a Brownian motion.
- We shall consider what happens when M is the martingale component of a Markov chain.

$$Q = Y_t - \int_{]t,T]} F(\omega, u, Y_{u-}, Z_u) du + \int_{]t,T]} Z_u dM_u$$

The key idea is that:

- Q is not known until time T
- The dynamics are given over the 'future' interval $]t, T]$.
- The solution is a pair (Y, Z), which is *adapted*, that is, Y_t and Z_t are known at time t.
- In fact, Z is a *predictable* (typically left-continuous) process.

A special case: Black-Scholes pricing

- Suppose we have a market with two assets: a stock S following a simple geometric Brownian motion, and a risk free Bond B.
- Assume zero interest rates (for simplicity).
- Let $Q = Y_T$ be a contingent payoff.
- Standard theory tells us that there is a unique 'no-arbitrage' price at time t,

$$Y_t = E_{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} Y_T \middle| \mathcal{F}_t\right].$$

- We can also write

$$Y_T = E_{\mathbb{P}}(Y_T|\mathcal{F}_t) + L_t^T$$

where $E_{\mathbb{P}}(Y_{t+1}|\mathcal{F}_t)$ is the (real-world) conditional mean value of Y_{t+1}, and L_t^T is an $\mathcal{F}_T$-measurable random variable with $\mathcal{F}_t$-conditional mean value zero.

- Using the Martingale Representation theorem, we can write

$$L_t^T = \int_{]t,T]} Z_u dM_u$$

where M is the underlying Brownian motion, and Z is a predictable process.

We can then do some basic algebra:

$$\begin{aligned} Y_T &= E_{\mathbb{P}}(Y_T|\mathcal{F}_t) + L_t^T \\ &= Y_t - E_{\mathbb{Q}}(Y_T|\mathcal{F}_t) + E_{\mathbb{P}}(Y_T|\mathcal{F}_t) + \int_{]t,T]} Z_u dM_u \\ &= Y_t - E_{\mathbb{Q}}(Y_T - E_{\mathbb{P}}(Y_T|\mathcal{F}_t)|\mathcal{F}_t) + \int_{]t,T]} Z_u dM_u \\ &= Y_t - E_{\mathbb{Q}}\left(\int_{]t,T]} Z_u dM_u \middle| \mathcal{F}_t\right) + \int_{]t,T]} Z_u dM_u. \end{aligned}$$

Through the use of Girsanov's Theorem, one can show

$$E_{\mathbb{Q}}\left(\int_{]t,T]} Z_u dM_u \middle| \mathcal{F}_t\right) = \int_{]t,T]} \theta_u Z_u du$$

for some process θ.
Hence the price of Y is simply the solution of

$$Y_T = Y_t - \int_{]t,T]} \theta_u Z_u du + \int_{]t,T]} Z_u dM_u$$

This is a special case of a BSDE, with

$$F(\omega, t, Y_t, Z_t) = \theta_t Z_t.$$

Markov Chains and Martingales

- Suppose we have an N state Markov chain X, with rate matrix A.
- We associate each of the states of X with the standard basis vectors in $\mathbb{R}^N$.
- We can write X as a semimartingale

$$X_t = X_0 + \int_{]0,t]} A_u X_{u-} du + M_u.$$

- $(A_u X_{u-})$ is a vector giving the rate, at time u, at which X will jump into each of the alternative states.

Existence of BSDE solutions

Theorem (Pardoux & Peng (1990))

Suppose M is an N-dimensional Brownian motion generating $\{\mathcal{F}_t\}$. If $Q \in L^2(\mathcal{F}_T)$, (Q has finite variance and is known at time T) and F is uniformly Lipschitz continuous $dt \times \mathbb{P}$-a.e., then there exists a unique solution (Y, Z) which is square integrable and adapted.

Theorem (Cohen & Elliott (2008))

This is also true when M is the martingale component of a Markov chain.

Uses of BSDEs

- Markov Chains have been used to model various market characteristics – particularly where regime switching is involved.
- BSDEs can be used to price assets defined in these markets.
- BSDEs can also be used for other things – pricing with constraints, recursive utilities, *risk theory*.

Comparing BSDEs

- For many of these applications, we need to be able to compare the solutions to BSDEs with different terminal conditions.
- When thinking about pricing, we need to ensure that the prices we give are Arbitrage free.
- When thinking about risk, we want a more positive outcome to be preferred to a less positive outcome
- The key result here is the *comparison theorem*.
- This is very closely related to the maximum principle for second-order nonlinear PDEs.

The Comparison Theorem

Consider two BSDEs. Suppose

(i) $Q^1 \geq Q^2$ a.s.

(ii) $F^1(\omega, u, Y^2_{u-}, Z^2_u) \geq F^2(\omega, u, Y^2_{u-}, Z^2_u)$ a.s. for almost all u.

(iii) There exists $\epsilon > 0$ such that, for all u, if

$$(e_j^* A_u X_{u-})[Z^1_u - Z^2_u](e_j - X_{u-}) \geq -\epsilon \|Z^1_u - Z^2_u\|_{X_{u-}}$$

for all e_j then

$$F^1(u, Y^2_{u-}, Z^1_u) - F^1(u, Y^2_{u-}, Z^2_u) \geq -[Z^1_u - Z^2_u]A_t X_{u-}$$

with equality only if $\|Z^1_u - Z^2_u\|_{X_{u-}} = 0$.

Then $Y^1_t \geq Y^2_t$ for all t.

The Comparison Theorem

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(i) $Q^1 \geq Q^2$ a.s.

(ii) $F^1(\omega, u, Y^2_{u-}, Z^2_u) \geq F^2(\omega, u, Y^2_{u-}, Z^2_u)$ a.s. for almost all u.

(iii) Either

1. All jumps of M have no impact on $Y^1 - Y^2$, or
2. One of the jumps of M should result in a (significant) decrease in $Y^1 - Y^2$, or
3. The difference between Z^1 and Z^2 should have a decreasing effect on $Y^1 - Y^2$ when there are no jumps.

Then $Y^1_t \geq Y^2_t$ for all t.

The Comparison Theorem

The intuitive version of this result is:

- Provided conditions (ii) and (iii) hold, if the terminal value Q^1 is more than the value Q^2 with probability one, then the initial value Y_0^1 must be more than Y_0^2.
- In terms of prices – if one asset is always worth more than another in the future, it must have a higher price today.

When the comparison fails

- When M is a Brownian motion, Assumption (iii) is not needed.
- There are various technical reasons for this – basically because we know that equivalent martingale measures exist for certain quantities.
- In our setting, Assumption (iii) is unavoidable. (In some cases it can be shown to be a necessary condition.)

When the comparison fails

Without (iii), we could have the situation where

- $Y_0^1 - Y_0^2$ starts just below zero,
- increases between jumps of M, and
- jumps up at each jump of M.

Then $Y_T^1 - Y_T^2 = Q^1 - Q^2$ ends above zero – so we have a situation where the terminal values are $Q^1 \geq Q^2$, but the initial values are $Y_0^1 \leq Y_0^2$.

Given this theory, we are now able to construct explicit examples of nonlinear expectations.

- These are the g-expectations in a Markov Chain context (cf. Peng and others for Brownian motion)
- We begin with a general definition.

Nonlinear Expectations

For some terminal time T, we define an '$\mathcal{F}_t$-consistent nonlinear expectation' $\mathcal{E}$ to be a family of operators

$$\mathcal{E}(\cdot|\mathcal{F}_t) : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t); t \leq T$$

with

1. (Monotonicity) If $Q^1 \geq Q^2$ $\mathbb{P}$-a.s.,

$$\mathcal{E}(Q^1|\mathcal{F}_t) \geq \mathcal{E}(Q^2|\mathcal{F}_t)$$

2. (Constants) For all $\mathcal{F}_t$-measurable Q,

$$\mathcal{E}(Q|\mathcal{F}_t) = Q$$

Nonlinear Expectations

3. (Recursivity) For $s \leq t$,

$$\mathcal{E}(\mathcal{E}(Q|\mathcal{F}_t)|\mathcal{F}_s) = \mathcal{E}(Q|\mathcal{F}_s)$$

4. (Regularity) For any $A \in \mathcal{F}_t$,

$$\mathcal{E}(I_A Q|\mathcal{F}_t) = I_A \mathcal{E}(Q|\mathcal{F}_t).$$

Nonlinear Expectations

Two further properties are commonly desirable

5. (Translation invariance) For any $q \in L^2(\mathcal{F}_t)$,

$$\mathcal{E}(Q + q|\mathcal{F}_t) = \mathcal{E}(Q|\mathcal{F}_t) + q.$$

6. (Concavity) For any $\lambda \in [0, 1]$,

$$\mathcal{E}(\lambda Q^1 + (1 - \lambda)Q^2|\mathcal{F}_t) \geq \lambda\mathcal{E}(Q^1|\mathcal{F}_t) + (1 - \lambda)\mathcal{E}(Q^2|\mathcal{F}_t)$$

Nonlinear Expectations

There is a relation between nonlinear expectations and convex risk measures:

- If (1)-(6) are satisfied, then for each t,

$$\rho_t(Q) := -\mathcal{E}(Q|\mathcal{F}_t)$$

  defines a dynamic convex risk measure. These risk measures are *time consistent*.
- Using risk measures related to BSDEs is less intuitive than Value at Risk, etc..., but gives better behaviour – we know when they are convex and time consistent.

Theorem

Let F be a driver for a BSDE. Suppose $F(\omega, t, Y_t, 0) = 0$ and F satisfies Assumption (iii) of the comparison theorem, then the solution to a BSDE is a nonlinear expectation, that is

$$\mathcal{E}(Q|\mathcal{F}_t) = Y_t$$

gives a nonlinear expectation satisfying Axioms (1-4).

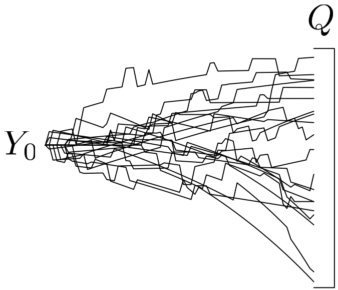
- If F does not depend on Y, then the nonlinear expectation is translation invariant, (Axiom 5), i.e. for all $q \in L^2(\mathcal{F}_t)$,

$$\mathcal{E}(Q + q|\mathcal{F}_t) = \mathcal{E}_t(Q|\mathcal{F}_t) + q.$$

- If F is concave, then $\mathcal{E}(\cdot|\mathcal{F}_t)$ is concave (Axiom 6), that is, the nonlinear expectation is *risk averse*.

Geometry

When F does not depend on Y and $F(\omega, u, Y_{u-}, 0) = 0$, then the values Y_t must lie between the possible values of Q, given information up to time t.



Geometry

- If we think of Y as the *discounted* price of an asset, this means that the current value of the asset must lie within the possible range of future values – if the value can increase, then it could also decrease.
- This result can be generalised to when Y is *undiscounted* but interest rates are deterministic, or, mathematically, when $F(\omega, t, Y, 0)$ is deterministic.
- The comparison theorem requires that this holds for the *difference* of two Y values.
- The details of these assumptions are critical when considering these equations on infinite horizons.

Conclusions

- The theory of BSDEs based on Markov chains allows us to develop dynamic risk measures/nonlinear expectations in this setting.
- A significant distinction between the case of Markov chain noise and Brownian noise is the requirements for the comparison theorem.
- Combining Brownian motion and Markov Chains into a single system (as in Regime switching models) would be a straightforward extension of this work.

Future work

- Extending the comparison theorem to other settings is nontrivial, but would help with developing a more general theory of BSDEs. (We have some results here.)
- Are all dynamic nonlinear expectations BSDE solutions?
  - In discrete time, yes;
  - With Brownian noise/general processes, yes, under certain boundedness assumptions (Coquet et al 2002, Hu et al 2008, Cohen 2012)
- Calculating the values of these solutions is of practical interest. This can be done using Monte-Carlo regression methods.