Stability and uniform fluctuation estimates of Ensemble Kalman-Bucy filters

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Synthesis joint works:

- AAP-17 (Unif. EnKBF)+ Tugaut.
- SIAM Control & Opt.-17 (Unif. En-EKBF)+ Kurtzmann, Tugaut.
- SIAM Control & Opt.-17/18 (Stability KBF)+ Bishop
- SPA-17 (Perturbation KB)+ Bishop, S. Pathiraja.
- Arxiv 1 (Stability EKBF)+ Kurtzmann, Tugaut.
- Arxiv 2 (Stochastic Riccati)+ Bishop, A. Niclas.
- Arxiv 3 (1d-case)+ Bishop, Kamatani, Rémillard.

Kalman-Bucy filter

Linear+Gaussian filtering problem

$$\left\{\begin{array}{lclcl} dX_t & = & A\ X_t\ dt & + & R^{1/2}\ dW_t \ \in \mathbb{R}^r \\ dY_t & = & C\ X_t\ dt & + & \Sigma^{1/2}\ dV_t \qquad \rightsquigarrow \mathcal{F}_t := \sigma(Y_s,\ s \leq t). \end{array}\right.$$

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Optimal $\mathbb{L}_2$-filter = Kalman-Bucy filter

$$\widehat{X}_t := \mathbb{E}(X_t \mid \mathcal{F}_t) \quad \text{and} \quad P_t := \mathbb{E}\left((X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))\,(X_t - \mathbb{E}(X_t \mid \mathcal{F}_t))'\right)$$

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State estimate

$$d\widehat{X}_t = A\; \widehat{X}_t\; dt + P_t\; C'\Sigma^{-1}\; \left(dY_t - C\widehat{X}_t\; dt\right)$$

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$$d\widehat{X}_t = A\,\widehat{X}_t\,dt + P_t\,C'\Sigma^{-1}\left(dY_t - C\widehat{X}_t\,dt\right)$$

with the gain given by the matrix Riccati equation

$$\partial_t P_t = \mathrm{Ricc}(P_t) := AP_t + P_tA' - P_t\mathbf{S}P_t + R \quad \text{with} \quad \mathbf{S} := \mathbf{C}'\boldsymbol{\Sigma}\mathbf{C}$$

Reformulation $\rightsquigarrow$ Nonlinear Kalman-Bucy diffusion

$\Longleftrightarrow$ **McKean-Vlasov type diffusions $\overline{X}_t$ such that**

$$\eta_t := \mathrm{Law}(\overline{X}_t \mid \mathcal{F}_t) \; = \; \mathcal{N}\left[\widehat{X}_t, P_t\right]$$

$\rightsquigarrow$ **Interacting with their conditional mean and covariance matrices**

$$\mathcal{P}_{\eta_t} = \eta_t\left[(e - \eta_t(e))(e - \eta_t(e))'\right] \quad \text{with} \quad e(x) := x.$$

3 classes of McKean-Vlasov type diffusions

1) "Vanilla EnKF" ($\rightsquigarrow$ discrete time - Evensen 94)

$$d\overline{X}_t = A\,\overline{X}_t dt + R^{1/2}\,d\overline{W}_t + \mathcal{P}_{\eta_t} C' \Sigma^{-1}\,\left[dY_t - \left(C\,\overline{X}_t dt + \Sigma^{1/2}\,d\overline{V}_t\right)\right]$$

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3) Pure transport equation (Reich-Cotter 13):

$$d\overline{X}_t = A\,\overline{X}_t dt$$

$$+\frac{1}{2}\left(R - \mathcal{P}_{\eta_t} S \mathcal{P}_{\eta_t}\right)\mathcal{P}_{\eta_t}^{-1}\left(\overline{X}_t - \eta_t(e)\right)dt + \mathcal{P}_{\eta_t} C'\Sigma^{-1}\left[dY_t - C\eta_t(e)dt\right]$$

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$\oplus$ **Many others, adding** $\mathcal{Q}_{\eta_t}\mathcal{P}_{\eta_t}^{-1}(\overline{X}_t - \eta_t(e))\, dt$ **for any** $\mathcal{Q}'_{\eta_t} = -\mathcal{Q}_{\eta_t}$.

The Ensemble Kalman-Bucy filter

(Case 1) Mean field interpretation $\rightsquigarrow$ $N+1$ interacting diffusions

$$d\xi_t^i = A\,\xi_t^i dt + R^{1/2} d\overline{W}_t^i + p_t C'\Sigma^{-1}\left[dY_t - \left(C\xi_t^i dt + \Sigma^{1/2}\, d\overline{V}_t^i\right)\right]$$

with the rescaled particle covariance matrices

$$p_t := \left(1+\frac{1}{N}\right)\ \mathcal{P}_{\eta_t^N} = \frac{1}{N}\sum_{1\leq i\leq N+1}\left(\xi_t^i - m_t\right)\left(\xi_t^i - m_t\right)'$$

and the empirical measures

$$\eta_t^N := \frac{1}{N+1}\sum_{1\leq i\leq N+1}\delta_{\xi_t^i} \quad \text{and the sample mean} \quad m_t := \frac{1}{N+1}\sum_{1\leq i\leq N+1}\xi_t^i.$$

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where is the Riccati equation?

Th1: The EnKF equations

The EnKF equation

$$dm_t = A\ m_t dt + p_t\ C'\Sigma^{-1}\ (dY_t - Cm_t\ dt) + \frac{1}{\sqrt{N+1}}\ d\overline{M}_t$$

with an **r**-dimensional martingale $\overline{M}_t = \left(\overline{M}_t(k)\right)_{1\leq k\leq r}$ with angle-brackets

$$\partial_t \langle \overline{M} \mid \otimes \mid \overline{M} \rangle_t = U + p_t V p_t.$$

With

1) $(U, V) = (R, S)$ 2) $(U, V) = (R, 0)$ 3) $(U, V) = (0, 0)$

The particle/ensemble Riccati equation

$$dp_t = \mathrm{Ricc}(p_t)\ dt + \frac{1}{\sqrt{N}}\ dM_t$$

Symmetric matrix-valued martingale $M_t = (M_t(k,l))_{1\leq k,l\leq r}$

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Angle brackets = the Wick-type formula $((\centerdot \otimes \centerdot)^\sharp :=$ **entrywise)**

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Orthogonality property

$$\forall 1 \leq k, l, l' \leq r \qquad \left\langle M(k,l), \overline{M}(l')\right\rangle_t = 0.$$

(Hyp.) Observability + Controllability

$$d\overline{X}_t = (\mathbf{A} - \mathbf{P_t S})\, \overline{X}_t dt \;+\; R^{1/2}\, d\overline{W}_t + P_t C' \Sigma^{-1} \left[dY_t - \Sigma^{1/2}\, d\overline{V}_t \right]$$

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Steady state: $\exists! P > 0$ such that $\mathrm{Ricc}(P) = 0$ and spectral abscissa

$$\varsigma(\mathbf{A} - \mathbf{PS}) := \max\{\mathrm{Re}(\lambda)\ :\ \lambda \in \mathrm{Spec}(A - PS)\} < 0$$

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STABLE EVEN WHEN A is unstable.

Bucy's theorems

Notation (Peano-Baker-expo-type series) - State transition matrix

$$\mathcal{E}_t = \exp\left[\oint_0^t Q_u \, du\right] \Longrightarrow \partial_t \mathcal{E}_t = Q_t \mathcal{E}_t$$

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Theo [Bucy] $\exists \upsilon, \alpha, \beta > 0$ and $\mathcal{G}_\upsilon^+ > \mathcal{G}_\upsilon^- > 0$: $\forall P_0$ and $t \geq \upsilon$

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Also true for any $t \geq 0$ with α depending on P_0.

Some consequences : $\psi_{s,t}(x, Q)$ = stochastic flow KB filter

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$$\mathbb{E}\left(\|\psi_{s,t}(x_1, Q_1) - \psi_{s,t}(x_2, Q_2)\|_2^n \mid X_s\right)^{1/n}$$

$$\leq c_{Q_1,Q_2}\ e^{-\nu(t-s)}\ \left[\|x_1 - x_2\|_2 + \left\{\|x_2 - X_s\|_2\ +\ \sqrt{n}\right\}\ \|Q_1 - Q_2\|_2\right]$$

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- **same estimate for $\overline{\psi}_{s,t}(x, Q)$ = stochastic flow Kalman-Bucy (nonlinear) diffusion**
- **similar estimate for $\phi_t(Q)$ = semigroup of the Riccati equation**

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Any choice

$$\nu \in \{(1-\epsilon)\ \varsigma(A - PS),\ \lambda_{max}((A - PS)_{sym})\}$$

is fine but c_{Q_1,Q_2} maybe larger than you expect.

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$$\Longrightarrow \quad \forall \epsilon > 0 \qquad \mathbb{E}(e^{\epsilon q_t}) = \infty \quad \text{and} \quad \forall m \geq N+2 \qquad \mathbb{E}(q_t^m) = \infty$$

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- **Case 3** = *Pure transport* $\rightsquigarrow$ simple estimates using Lipschitz and contraction inequalities w.r.t. initial conditions

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$$(m_t, X_t, \mathbf{p_t}) = (\text{sample mean}, \text{true signal}, \textbf{sample covariance})$$

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$$d(m_t - X_t) = (A - \mathbf{p_t} S)\,(m_t - X_t)\,dt + p_t\; C'\Sigma^{-1/2}\; dV_t + \frac{\mathbf{d\overline{M}_t}}{\sqrt{\mathbf{N+1}}}$$

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Observations:

- **Time varying** $\oplus$ **stochastic type** Ornstein-Ulhenbeck diffusion

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- The matrix $(A - pS)$ may be ill-conditioned in the sense that

  $$\exists p\ :\ \lambda_{max}((A - pS)_{sym}) > 0 > \lambda_{min}((A - pS)_{sym})$$

Multivariate case

Hyp 1: $S > 0 \rightsquigarrow$ **up a change of basis**

$$(A, S) \rightsquigarrow (\overline{A}, \overline{S}) := (S^{1/2} A S^{-1/2}, I) \quad \rightsquigarrow \quad + \text{Hyp } 2 \quad \mu(\overline{A}) < 0$$

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Theo 1 [+Tugaut AAP-17] [Hyp 1 + Hyp 2] $\forall n \geq 1 \; \exists N_n \geq 1$:

$$N \geq N_n \Longrightarrow \sup_{t \geq 0} \left[\mathbb{E}\left(\| p_t - P_t \|^n \right) \vee \mathbb{E}(\| m_t - \widehat{X}_t \|^n) \right] < c/\sqrt{N}$$

for the spectral of the Frobenius norm.

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Theo 2 [+Bishop et al. 17] [only Observability + Controllability]

Uniform Riccati estimates even if we have the under-bias estimate

$$\forall t > 0 \qquad 0 < p_t \quad \text{but} \quad 0 < \mathbb{E}(p_t) \leq P_t$$

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$$\Downarrow$$

Theo 1 [+Tugaut AAP-17] [Hyp 1 + Hyp 2] $\forall n \geq 1 \; \exists N_n \geq 1$:

$$N \geq N_n \Longrightarrow \sup_{t \geq 0} \left[\mathbb{E}\left(\|p_t - P_t\|^n \right) \vee \mathbb{E}(\|m_t - \widehat{X}_t\|^n) \right] < c/\sqrt{N}$$

for the spectral of the Frobenius norm.

Theo 2 [+Bishop et al. 17] [only Observability + Controllability]

Uniform Riccati estimates even if we have the under-bias estimate

$$\forall t > 0 \qquad 0 < p_t \quad \text{but} \quad 0 < \mathbb{E}(p_t) \leq P_t$$

+ Uniform CLT rates+ Bias-Taylor type expansions + Robustness and Perturbations analysis (inflation, masking, shrinkage, projections,...),...

1d $\rightsquigarrow$ Closed form Riccati semigroups/tangent proc.

Deterministic Riccati flow $\phi_t(Q)$ on $\mathbb{R}_+$: $\mathrm{Ricc}(\varpi_\pm) = 0$ for

$$S\varpi_- := A - \lambda/2 < 0 < S\varpi_+ := A + \lambda/2$$

with

$$\lambda = 2\sqrt{A^2 + RS}$$

$$\Downarrow$$

$$\sup_{Q \geq 0} \left[|\phi_t(Q) - \varpi_+| \vee exp\left(2\int_0^t [A - \phi_s(Q)S]ds\right)\right] \leq c \ \exp\left(-\lambda t\right)$$

Reversible measures

Stochastic Riccati flow $Q_t := \Phi_t^\epsilon(Q_0) \in \mathbb{R}_+$ **with** $\epsilon = 2/\sqrt{N}$**:**

$$dQ_t \stackrel{law}{=} \mathrm{Ricc}(Q_t)dt + \epsilon\ \sqrt{Q(U+QVQ)}\ dW_t$$

$$\text{with} \quad \epsilon^2 U < 2R \quad \Longrightarrow \quad \text{origin repellent}$$

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Reversible measures $\pi_\epsilon(dx)$ **on** $\mathbb{R}_+$**:**

- $U \wedge V > 0 \rightsquigarrow$ Heavy tails

$$\propto\; 1_{\mathbb{R}_+}(x)\, \frac{x^{\frac{2}{\epsilon^2}\frac{R}{U}-1}}{[U+Vx^2]^{1+\frac{1}{\epsilon^2}\left(\frac{R}{U}+\frac{S}{V}\right)}}\; \exp\left[\frac{4}{\epsilon^2}\,\frac{A}{\sqrt{UV}}\; \tan^{-1}\left(x\sqrt{\frac{V}{U}}\right)\right] dx.$$

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- $U > V = 0 \rightsquigarrow$ Gaussian-type tails

$$\propto\ 1_{\mathbb{R}_+}(x)\ x^{\frac{2}{\epsilon^2}\frac{R}{U}-1}\ \exp\left[-\frac{S}{U\epsilon^2}\ \left(x-2\ \frac{A}{S}\right)^2\right]\ dx.$$

Stability Markov transition semigroup P_t^ϵ

Theo [+Bishop, Kamatani, Rémillard 17] $\forall A$ and $\forall R \wedge S > 0$

- $\exists \zeta, \epsilon_0 > 0$ and some Wasserstein distance $\mathbb{D}$ s.t. for any $0 \leq \epsilon \leq \epsilon_0$

$$\mathbb{D}(\mu_1 P_t^\epsilon, \mu_2 P_t^\epsilon) \leq \exp\left(-\lambda\ (1-\epsilon^2\zeta)\ t\right)\ \ \mathbb{D}(\mu_1, \mu_2)$$

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Some extensions
Case 2: If $A < 0$ or $0 < A < \sqrt{RS} \rightsquigarrow$ Poincaré inequalities (and $\mathbb{L}_2(\pi_\epsilon)$-contractions), ...

Consequences
Uniform estimates for En-KF + particle Riccati diffusions,...

Nonlinear models

Extended Kalman-Bucy-filters

$$d\widehat{X}_t = A(\widehat{X}_t)\ dt + P_t C'\ \Sigma^{-1}\ \left[dY_t - C\widehat{X}_t\ dt\right]$$

with the "stochastic" Riccati equation:

$$\partial_t P_t = \partial A(\widehat{X}_t) P_t + P_t\ \partial A(\widehat{X}_t)' + R - P_t S P_t$$

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McKean-Vlasov interpretation

$$\begin{aligned} d\overline{X}_t &= \mathcal{A}\left(\overline{X}_t, \eta_t(e)\right)\ dt + R^{1/2}\ d\overline{W}_t \\ &\qquad + \mathcal{P}_{\eta_t} C' R_2^{-1}\ \left[dY_t - \left(C\overline{X}_t\ dt + \Sigma^{1/2}\ d\overline{V}_t\right)\right] \end{aligned}$$

with the drift function

$$\mathcal{A}(x,m) := A[m] + \partial A[m]\ (x-m).$$

Extended Ensemble Kalman-Bucy-filters

En-EKF = Mean field particle model

$$d\xi_t^i = \mathcal{A}(\xi_t^i, m_t)\ dt + R^{1/2} d\overline{W}_t^i + p_t C' \Sigma^{-1} \left[dY_t - \left(C\xi_t^i\ dt + \Sigma^{1/2}\ d\overline{V}_t^i \right) \right]$$

with the sample means m_t and covariance matrices p_t and the drift

$$\mathcal{A}(\xi_t^i, m_t) := A[m_t] + \underbrace{\partial A[m_t]\ (\xi_t^i - m_t)}_{\text{Repulsion/Attraction w.r.t. } m_t}$$

Some illustrations

Langevin type signal processes

$$R = \sigma^2 \; Id \quad \text{and} \quad (A, \partial A) = (-\partial \mathcal{V}, -\partial^2 \mathcal{V})$$

Non quadratic potential $(q \in \mathbb{R}^r, \mathcal{Q}_1, \mathcal{Q}_2 \geq 0)$

$$\mathcal{V}(x) = \frac{1}{2} \langle \mathcal{Q}_1 x, x \rangle + \langle q, x \rangle + \frac{1}{3} \langle \mathcal{Q}_2 x, x \rangle^{3/2}$$

Interacting diffusion gradient flows

$$\mathcal{V}(x) = \sum_{1 \leq i \leq r} \mathcal{U}_1(x_i) + \sum_{1 \leq i \neq j \leq r} \mathcal{U}_2(x_i, x_j)$$

for some convex confining potential $\mathcal{U}_i : \mathbb{R}^i \mapsto [0, \infty[$

Regularity conditions

Full observation $S = s\ Id$ and

$$
\begin{aligned}
-\lambda_{\partial A} &:= \sup_{x\in\mathbb{R}^r} \lambda_{max}(\partial A(x) + \partial A(x)') < 0 \\
\|\partial A(x) - \partial A(y)\| &\leq \kappa_{\partial A}\ \|x - y\|
\end{aligned}
$$

Examples: Langevin signal-diffusion

$$
(\lambda_{\partial A}, \kappa_{\partial A}) = \beta\ \left(2^{-1}\lambda_{min}(\mathcal{Q}_1),\ 2\lambda_{max}^{3/2}(\mathcal{Q}_2)\right).
$$

more generally $\partial^2\mathcal{V} \geq v\ Id \oplus$ *Lipschitz condition*

Stability theorem

$$(\overline{X}_t, \overline{Z}_t) := \text{McKean-Vlasov starting at} \quad (\overline{X}_0, \overline{Z}_0)$$

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$$\Downarrow$$

Theo [+Kurtzmann-Tugaut]

When $\lambda_{\partial A}$ is sufficiently large we have

$$\mathbb{W}_2(\text{Law}(\overline{X}_t), \text{Law}(\overline{Z}_t)) \leq c \ \exp\left[-t \ \lambda\right] \quad \text{for some} \quad \lambda > 0.$$

$\exists$ *more explicit description in terms of* $(R, S, \kappa_{\partial A})$.

Propagation of chaos

$$\mathbb{P}_t^N := \mathrm{Law}(m_t, p_t) \qquad \mathbb{P}_t := \mathrm{Law}(\widehat{X}_t, P_t)$$

and

$$\mathbb{Q}_t^N := \mathrm{Law}(\xi_t^1) \qquad \mathbb{Q}_t := \mathrm{Law}(\overline{X}_t)$$

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$$\Downarrow$$

Theo [+Kurtzmann-Tugaut]

When $\lambda_{\partial A}$ is sufficiently large, $\exists \beta \in]0, 1/2]$ s.t.

$$\sup_{t \geq 0} \mathbb{W}_2 \left(\mathbb{P}_t^N, \mathbb{P}_t \right) \vee \sup_{t \geq 0} \mathbb{W}_2 \left(\mathbb{Q}_t^N, \mathbb{Q}_t \right) \leq c \; N^{-\beta}$$

as soon as $\mathrm{tr}(P_0)$ is not too large and N large enough...