Market Price of Longevity Risk for A Multi-Cohort Mortality Model with Application to Longevity Bond Option Pricing

Yajing Xu, Michael Sherris and Jonathan Ziveyi

School of Risk & Actuarial Studies, UNSW Business School, UNSW Sydney
ARC Centre of Excellence in Population Ageing Research (CEPAR)

7 December 2017

Workshop: Risk: Modelling, Optimization and Inference with applications in Finance, Insurance and Superannuation
UNSW Sydney

Introduction

Application of stochastic mortality models to risk management of pension and annuity funds - a combination of financial and insurance methods.

Major challenges in developing a traded market in longevity risk and innovation in financial hedging instruments including S-forwards, q-forwards, swaps and options.

Difficulty in calibrating models because of lack of liquid traded securities to determine risk neutral dynamics for pricing and risk management.

Common approach is to calibrate a model to historical mortality data and imply market price of risk from mortality linked products such as life annuities or longevity swaps.

Our approach is to apply a multi-cohort affine mortality model and calibrate prices of risk using Blackrock's CORI with applications to options on longevity linked bonds.

Blackrock CORI - from https://www.blackrock.com/cori/what-is-cori

Estimated annual retirement income is the amount of money your savings may generate every year of your retirement starting at age 65 (or starting today if you are 65 or over), lasting as long as you live, and including a cost-of-living adjustment.

Risk: The promise to give you $1 in the future involves investment risk. Current market information about what insurers are charging to manage similar risk impacts your CoRI Index level.

Interest Rates: Interest rate changes can cause daily fluctuations in your CoRI Index level.

Life Expectancy: Because we won't all live to 115, the CoRI Index level is reduced using actuarial calculations similar to ones used by Social Security, pension plans and insurance companies.

As with any index, you can't invest in the index directly, but you can invest in a fund that tracks the index.

Blackrock CORI

BlackRock introduced the CoRI Indexes in June 2013 to help investors estimate and track the cost of $1 of annual lifetime income at retirement.

The CoRI consists of twenty indexes corresponding to twenty cohorts born from 1941 to 1960 in U.S.

For cohorts with an age below 65 the index is the discounted cost of purchasing inflation-adjusted lifetime retirement income at age 65, and for other cohorts it is the cost of purchasing inflation-adjusted retirement income for remaining life.

The CoRI indexes are constructed based on real-time market data, do not include any fees or premium taxes that would be associated with the price of an annuity.

Investors can use the CoRI index as a risk metric directly or invest in the BlackRock CoRI Funds that track the index.

Longevity Index

Value of a longevity index is the discounted value of lifetime annual income of \$1 for cohort i - a sum of longevity zero coupon bond prices.

$$I_x^i(t) = \sum_{j=1}^{x^*-x} \bar{P}_x^i(t, t+j), \tag{1}$$

where x^* is the maximum age.

Longevity zero-coupon bond $\bar{P}_x^i(t, T)$, pays 1 at time T if the cohort i is alive at time T.

Similarity to defaultable zero-coupon bond (as defined in Schönbucher (2003)), where the mortality intensity corresponds to the default intensity.

Longevity Index

Price of a longevity zero-coupon bond, in terms of zero coupon bond price and risk neutral survival probability

$$\begin{aligned}\bar{P}^i_x(t,T) &= E^Q\left[\mathrm{e}^{-\int_t^T \left(r(s)+\mu^i(x,s)\right)ds}|\mathcal{F}(t)\right] \\ &= E^Q\left[\mathrm{e}^{-\int_t^T r(s)ds}|\mathcal{G}(t)\right] E^Q\left[\mathrm{e}^{-\int_t^T \mu^i(x,s)ds}|\mathcal{H}(t)\right] \\ &= P(t,T)S^{Q,i}(x,t,T), \qquad (2)\end{aligned}$$

assuming dynamics of mortality rates and interest rates are independent.

$P(t,T)$ zero coupon bond maturing at time T and $S^{Q,i}(x,t,T)$ is risk neutral survival probability to time T.

Blackrock CORI

Assuming dynamics of mortality rates are independent of that of interest rates, for $i = 1951, ..., 1960$, we have

$$\begin{aligned} I_x^{CoRI,i}(0) &= \sum_{j=1}^{x^*-65} \bar{P}_x^i(0, 65-x+j) \\ &= \sum_{j=1}^{x^*-65} P(0, 65-x+j) S^{Q,i}(x, 0, 65-x+j), \end{aligned} \tag{3}$$

and for $i = 1941, ..., 1950$,

$$\begin{aligned} I_x^{CoRI,i}(0) &= \sum_{j=0}^{x^*-x} \bar{P}_x^i(0, j) \\ &= \sum_{j=0}^{x^*-x} P(0, j) S^{Q,i}(x, 0, j), \end{aligned} \tag{4}$$

where x^* is the maximum age, and is set to 115 by BlackRock.

Mortality Model - Continuous Time Dynamics

Multi-cohort mortality model developed by Xu et al. (2015). Three-factor affine mortality model, mortality intensity process for each cohort i aged $x + t$ at time t is

$$\mu^i(x,t) = X_1(t) + X_2(t) + Z^i(t), \tag{5}$$

where $X_1(t)$, $X_2(t)$ are two common factors and $Z^i(t)$ is the cohort specific factor.

Under best-estimate measure $\bar{Q}$, the state variables $(X_1(t), X_2(t), Z^i(t))$ have the following dynamics

$$dX_j(t) = -\phi_j X_j(t)dt + \sigma_j dW_j^{\bar{Q}}(t), j = 1, 2 \tag{6}$$

$$dZ^i(t) = -\phi_3^i Z^i(t)dt + \sigma_3^i dW_3^{\bar{Q},i}(t) \tag{7}$$

where ϕ_1, ϕ_2, ϕ_3^i, σ_1, σ_2 and σ_3^i are constants, and $W_1^{\bar{Q}}(t)$, $W_2^{\bar{Q}}(t)$ and $W_3^{\bar{Q},i}(t)$ are standard Wiener processes under $\bar{Q}$.

Mortality Model - Survival Probability

Best-estimate survival probability, $S^{\bar{Q},i}(x, t, T)$ for cohort i aged x at time t over duration $T - t$, has closed-form solution:

$$S^{\bar{Q},i}(x,t,T) = E^{\bar{Q}}[e^{-\int_t^T \mu^i(x,s)ds} | \mathcal{F}_t]$$
$$= e^{B_1(t,T)X_1(t)+B_2(t,T)X_2(t)+B_3^i(t,T)Z^i(t)+A^i(t,T)}, \qquad (8)$$

where

$$B_1(t,T) = -\frac{1-e^{-\phi_1(T-t)}}{\phi_1},$$
$$B_2(t,T) = -\frac{1-e^{-\phi_2(T-t)}}{\phi_2},$$
$$B_3^i(t,T) = -\frac{1-e^{-\phi_3^i(T-t)}}{\phi_3^i},$$
$$A^i(t,T) = \frac{1}{2}\sum_{j=1}^{2}\frac{\sigma_j^2}{\phi_j^3}\left[\frac{1}{2}(1-e^{-2\phi_j(T-t)}) - 2(1-e^{-\phi_j(T-t)}) + \phi_j(T-t)\right]$$

Mortality Model - Price of Risk

Best-estimate measure $\bar{Q}$ and the parameters are estimated using observed mortality data.

Change to the pricing risk-neutral measure Q - affine market price of risk specification in Dai and Singleton (2000) and Duffee (2002).

$\Lambda^i = (\lambda_{\mu,1}, \lambda_{\mu,2}, \lambda^i_{\mu,3})^T$ is vector of market prices of risk associated with cohort i

Market prices of longevity risk, $\lambda_{\mu,1}$ and $\lambda_{\mu,2}$ assumed the same across cohorts - common factors, but $\lambda^i_{\mu,3}$ differs by cohort.

From Girsanov's Theorem

$$dW_j^Q(t) = dW_j^{\bar{Q}}(t) + \lambda_{\mu,j}dt, j = 1, 2 \tag{10}$$

$$dW_3^{Q,i}(t) = dW_3^{\bar{Q},i}(t) + \lambda^i_{\mu,3}dt \tag{11}$$

where $W_1^Q(t)$, $W_2^Q(t)$ and $W_3^{Q,i}(t)$ are standard Wiener processes under the risk-neutral measure Q.

Mortality Model - Risk Neutral Survival Probability

Dynamics of mortality intensity under the risk-neutral measure Q

$$d\mu^i(x,t) = \left[-\phi_1 X_1(t) - \phi_2 X_2(t) - \phi_3^i Z^i(t) - \sigma_1\lambda_{\mu,1} - \sigma_2\lambda_{\mu,2} - \sigma_3^i\lambda_{\mu,3}^i\right] dt + \sigma_1 dW_1^Q(t) + \sigma_2 dW_2^Q(t) + \sigma_3^i dW_3^{Q,i}(t). \quad (12)$$

Risk-neutral survival probability

$$\begin{aligned} S^{Q,i}(x,t,T) &= E^Q\left[\mathrm{e}^{-\int_t^T \mu^i(x,s)ds} | \mathcal{H}(t)\right] \\ &= \mathrm{e}^{B_1(t,T)X_1(t)+B_2(t,T)X_2(t)+B_3^i(t,T)Z^i(t)+A^i(t,T)+C^i(t,T)} \\ &= S^{\bar{Q},i}(x,t,T)\mathrm{e}^{C^i(t,T)}, \end{aligned} \quad (13)$$

where

$$C^i(t,T) = \sum_{j=1}^{2} \frac{\sigma_j\lambda_{\mu,j}}{\phi_j^2}\left[\phi_j(T-t) - \left(1 - \mathrm{e}^{-\phi_j(T-t)}\right)\right] + \frac{\sigma_3^i\lambda_{\mu,3}^i}{(\phi_3^i)^2}\left[\phi_3^i(T-t) - \left(1 - \mathrm{e}^{-\phi_3^i(T-t)}\right)\right].$$

Mortality Model Estimation

U.S. male mortality data from Human Mortality Database for 1934 to 2013, aged 50 to 100, cohorts born 1884 to 1913.

Restructured on a cohort basis.

Sample survival probability for cohort i aged x at time t over duration $T - t$ is

$$\tilde{S}^i(x, t, T) = \prod_{s=1}^{T-t} (1 - \tilde{q}_x^i(t + s - 1)), \tag{14}$$

where $\tilde{q}_x^i(t)$ is the observed death rate at time t.

Corresponding sample average force of mortality is

$$\tilde{\mu}^i(x, t, T) = -\frac{1}{T - t} \log \tilde{S}^i(x, t, T). \tag{15}$$

Mortality Model Estimation

Figure 1 shows the average force of mortality in U.S. for cohorts born between 1884 and 1913, ages 50 to 100.

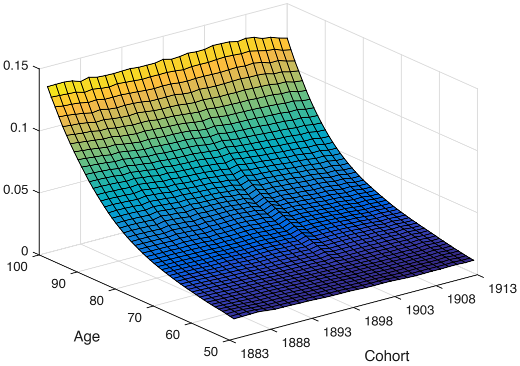

Figure 1: Male average force of mortality in U.S. for cohorts born between 1884 and 1913, ages 50 to 100.

Mortality Model Estimation

Two stage estimation for parameters - Kalman filter with the cohort data for (age-period) factors $X_1(t)$ and $X_2(t)$ and cohort factors by minimising residual error. For age-period factors the measurement equation is

$$y_t = -BX_t - A + \varepsilon_t, \qquad \varepsilon_t \sim N(0, H), \tag{16}$$

where

$$B = -\begin{pmatrix} \frac{1-e^{-\phi_1}}{\phi_1} & \frac{1-e^{-\phi_2}}{\phi_2} \\ \frac{1-e^{-2\phi_1}}{2\phi_1} & \frac{1-e^{-2\phi_2}}{2\phi_2} \\ \vdots & \vdots \\ \frac{1-e^{-n\phi_1}}{n\phi_1} & \frac{1-e^{-n\phi_2}}{n\phi_2} \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{1}{2}\sum_{i=1}^{2} \frac{\sigma_i^2}{\phi_i^3}[\frac{1}{2}(1-e^{-2\phi_i}) - 2(1-e^{-\phi_i}) + \phi_i] \\ \frac{1}{2}\sum_{i=1}^{2} \frac{\sigma_i^2}{2\phi_i^3}[\frac{1}{2}(1-e^{-4\phi_i}) - 2(1-e^{-2\phi_i}) + 2\phi_i] \\ \vdots \\ \frac{1}{2}\sum_{i=1}^{2} \frac{\sigma_i^2}{n\phi_i^3}[\frac{1}{2}(1-e^{-2n\phi_i}) - 2(1-e^{-n\phi_i}) + n\phi_i] \end{pmatrix},$$

Mortality Model Estimation

H is the covariance matrix for the Gaussian observation noise.

Volatility of the measurement error varies with age (Poisson variation) - assume H to be an n-dimensional diagonal matrix with elements $\sigma_{\varepsilon}^2(i)$ $(i = 1, 2, ..., n)$ taking an exponential form,

$$\sigma_{\varepsilon}^2(i) = \varepsilon_1 \exp(\varepsilon_2 i), \tag{17}$$

where ε_1 and ε_2 are two constants.

Volatility of the measurement error is exponentially increasing with age - reflecting empirical data.

Mortality Model Estimation

State variables evolve according to the transition equation

$$X_t = \Phi X_{t-1} + \eta_t, \qquad \eta_t \sim N(0, Q), \tag{18}$$

where $\Phi = \begin{pmatrix} e^{-\phi_1} & 0 \\ 0 & e^{-\phi_2} \end{pmatrix}$,

$$Q = \begin{pmatrix} \frac{\sigma_1^2}{2\phi_1}(1 - e^{-2\phi_1}) & 0 \\ 0 & \frac{\sigma_2^2}{2\phi_2}(1 - e^{-2\phi_2}) \end{pmatrix}.$$

Mortality Model Estimation

Kalman filter estimation results for the two common factors shown in Table 1.

ϕ_1	-0.14313
ϕ_2	-0.07904
σ_1	0.00006
σ_2	0.00018
$\varepsilon_1(\times 10^7)$	2.74881
$\varepsilon_2(\times 10^7)$	1.99699
Log likelihood	24440
RMSE	0.00051

Table 1: Kalman filter parameter estimates, log likelihood and RMSE.

Parameters for the cohort factors estimated by minimising calibration error in the estimated age-period model after including the cohort factor.

Grouping by 10 cohorts.

Estimation of cohort parameters shown in Table 2.

i cohort	ϕ_3^i	σ_3^i	Z^i
1884-1893	0.06791	0.00558	0.00163
1894-1903	0.05228	0.00719	0.00106
1904-1913	0.05463	0.00122	-0.00079

Table 2: Estimation results for cohort specific factors with a 10-year interval.

Mortality Model Estimation

Figure 2 shows mean absolute percentage error (MAPE) by age for the estimated survival probabilities.

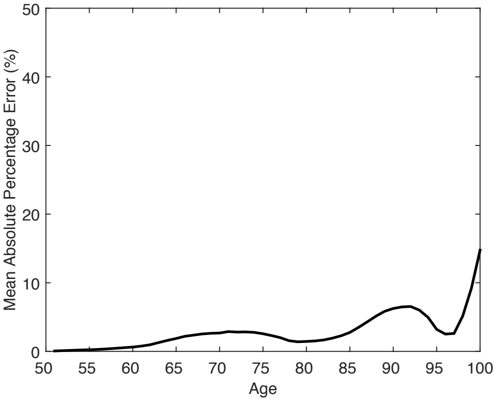

Figure 2: Mean percentage error of estimated survival probabilities.

Interest Rate Model

Arbitrage-free Nelson-Siegel (AFNS) Nelson and Siegel (1987) model developed by Christensen et al. (2011) - good empirical fit and arbitrage-free property Diebold and Li (2006)

A time-invariant yield-adjustment term required to make the dynamic Nelson-Siegel (DNS) model arbitrage-free.

The AFNS model combines the DNS factor loading structure and the arbitrage-free property of an affine term structure model.

Use the independent-factor AFNS model since it outperforms the correlated-factor AFNS model in out-of-sample forecasts (Christensen et al., 2011).

Interest Rate Model

$P(t, T)$ denotes the price of a discount bond with maturity of $T - t$, and $y(t, T)$ is the continuously compounded yield to maturity.

$$P(t, T) = \mathrm{e}^{-(T-t)y(t,T)}. \tag{19}$$

Christensen et al. (2011) propose the following representation for the yield function,

$$y(t, T) = L(t) + \frac{1 - e^{-\lambda(T-t)}}{\lambda(T-t)} S(t) + \left[\frac{1 - e^{-\lambda(T-t)}}{\lambda(T-t)} - e^{-\lambda(T-t)} \right] C(t) - \frac{V(t, T)}{T - t}, \tag{20}$$

where λ is the Nelson-Siegel parameter, and $-\frac{V(t,T)}{T-t}$ is a yield-adjustment term.

$L(t)$, $S(t)$ and $C(t)$ are the time-varying level, slope and curvature factors.

Dynamics under the risk-neutral Q-measure

$$\begin{pmatrix} dL(t) \\ dS(t) \\ dC(t) \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} L(t) \\ S(t) \\ C(t) \end{pmatrix} dt + \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix} \begin{pmatrix} d\tilde{W}_1^Q(t) \\ d\tilde{W}_2^Q(t) \\ d\tilde{W}_3^Q(t) \end{pmatrix}, \tag{21}$$

Under the real-world probability measure

$$\begin{pmatrix} dL(t) \\ dS(t) \\ dC(t) \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix} \left[\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} - \begin{pmatrix} L(t) \\ S(t) \\ C(t) \end{pmatrix} \right] dt + \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix} \begin{pmatrix} d\tilde{W}_1^P(t) \\ d\tilde{W}_2^P(t) \\ d\tilde{W}_3^P(t) \end{pmatrix}, \tag{22}$$

where κ_1, κ_2, κ_3, θ_1, θ_2 and θ_3 are real-world parameters.

Interest Rate Model

In the independent-factor case the yield-adjustment term is

$$
\begin{aligned}
-\frac{V(t,T)}{T-t} =& -\frac{(T-t)^2}{6}(s_1)^2 \\
& - \left[\frac{1}{2(\lambda)^2} - \frac{1}{(\lambda)^3}\frac{1-e^{-\lambda(T-t)}}{(T-t)} + \frac{1}{4(\lambda)^3}\frac{1-e^{-2\lambda(T-t)}}{(T-t)}\right](s_2)^2 \\
& - \left[\frac{1}{2(\lambda)^2} + \frac{1}{(\lambda)^2}e^{-\lambda(T-t)} - \frac{1}{4\lambda}(T-t)e^{-2\lambda(T-t)} - \frac{3}{4(\lambda)^2}e^{-\lambda(T-t)}\right. \\
& \left. - \frac{2}{(\lambda)^3}\frac{1-e^{-\lambda(T-t)}}{(T-t)} + \frac{5}{8(\lambda)^3}\frac{1-e^{-2\lambda(T-t)}}{(T-t)}\right](s_3)^2,
\end{aligned}
$$

where s_1, s_2 and s_3 are the volatility parameters.

Interest Rate Model - Estimation

Use end-of-month observations for real yields on Treasury Inflation Protected Securities (TIPS) interpolated by the U.S. Treasury.

TIPS are indexed to inflation as given by the Consumer Price Index (CPI) so provide a real rate of return.

Until the end of January 2010, the U.S. Treasury issued TIPS at fixed maturities, 5, 7, 10 and 20 years.

On February 22, 2010, a new TIP security with a maturity time of 30 years was introduced.

Use the Treasury real yield curve rates at 5 maturities of 5, 7, 10 20, and 30 years from February 2010 to March 2015.

Interest Rate Model - Estimation

Figure 3 shows the monthly yield curve rates

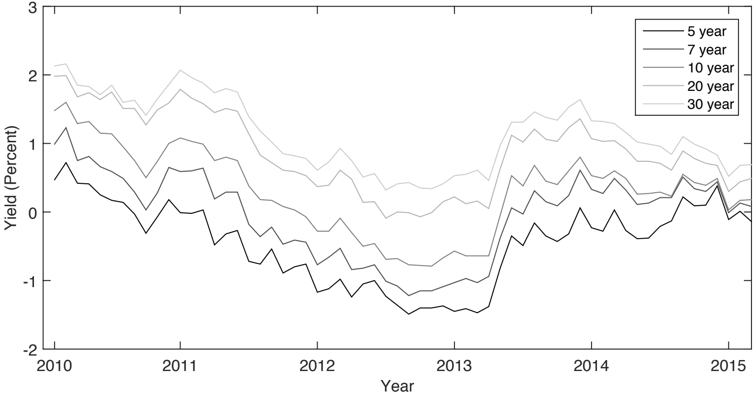

Figure 3: Time series of U.S. real yield curve rates, from February 2010 to March 2015.

Interest Rate Model - Estimation

Table 3 presents corresponding descriptive statistics.

Maturity	Mean	Std. Dev.	Min.	Max.	$\hat{\rho}(1)$	$\hat{\rho}(6)$	$\hat{\rho}(12)$
5Y	-0.4497	0.5955	-1.49	0.72	0.9037	0.5825	0.1160
7Y	-0.0461	0.6281	-1.22	1.23	0.9163	0.5814	0.0607
10Y	0.2984	0.6245	-0.79	1.60	0.9204	0.5568	0.0619
20Y	0.8790	0.5834	-0.09	1.99	0.9180	0.5715	0.0356
30Y	1.1452	0.5284	0.32	2.16	0.9137	0.5183	0.0094

Table 3: Descriptive statistics of U.S. real yield curve rates, $\hat{\rho}(i)$ denotes the sample autocorrelation with a time-lag of i months.

Interest Rate Model - Estimation

The AFNS model is represented in state-space form and estimated using a Kalman filter algorithm. The measurement equation is

$$y_t = -BY_t - A + \varepsilon_t, \qquad \varepsilon_t \sim N(0, H), \tag{23}$$

where

$$y_t = \begin{pmatrix} y_t(\tau_1) \\ \vdots \\ y_t(\tau_k) \end{pmatrix}$$

$$B = -\begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_k}}{\lambda\tau_k} & \frac{1-e^{-\lambda\tau_k}}{\lambda\tau_k} - e^{-\lambda\tau_k} \end{pmatrix}$$

$$Y_t = \begin{pmatrix} L(t) \\ S(t) \\ C(t) \end{pmatrix}, \quad A = \begin{pmatrix} \frac{V(\tau_1)}{\tau_1} \\ \vdots \\ \frac{V(\tau_k}{\tau_k} \end{pmatrix}.$$

Interest Rate Model - Estimation

The state transition equation is

$$Y_t = (I - e^{-K\Delta t})\Theta + e^{-K\Delta t} Y_{t-1} + \eta_t, \qquad \eta_t \sim N(0, Q), \tag{24}$$

where

$$K = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix},$$

and $Q = \int_0^{\Delta t} e^{-Ks} \Sigma \Sigma^\top e^{-(K^\top s)} ds$ with

$$\Sigma = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{pmatrix}.$$

Use monthly data with $\Delta t = \frac{1}{12}$.

Interest Rate Model - Estimation

Estimates for the independent-factor AFNS model are given in Table 4.

i	κ_i	θ_i	s_i
1	0.0458	0.0723	0.0061
2	0.1958	-0.0231	0.0047
3	1.2237	-0.0134	0.0059

Table 4: Parameter estimates for independent-factor AFNS model. The estimated λ is 0.7204, and the maximized log likelihood is 1622.75.

For the in-sample fit, residual means and their root mean square errors (RMSEs) are shown in Table 5.

Maturity	Mean	RMSE
5Y	-2.16	6.76
7Y	1.28	7.87
10Y	2.46	6.11
20Y	6.97	5.58
30Y	-8.69	6.36

Table 5: Residual means and root mean square errors for maturities in years. Means and RMSE's are in basis points.

Interest Rate Model - Estimation

As a measure of goodness-of-fit, Figure 4 plots the mean fitted curve for the independent-factor AFNS model.

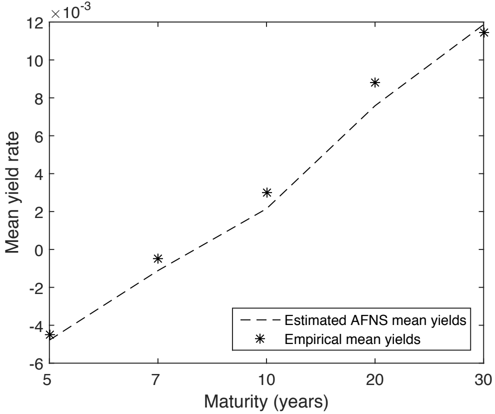

Figure 4: Empirical mean yield curve and the fitted AFNS mean yield curve, average from February 2010 to March 2015. Mean yields are in decimals.

Interest Rate Model - Estimation

Figure 5 shows the forecast yield curve at the end of March 2015.

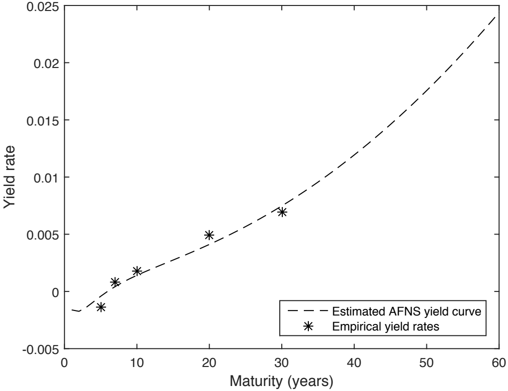

Figure 5: Empirical yield rates and the fitted AFNS yield curve at the end of March 2015. Yields are in decimals.

Implied Price of Longevity Risk

To estimate $\lambda_{\mu,1}$, $\lambda_{\mu,2}$, $\lambda^{i}_{\mu,3}$ ($i = 1941, ..., 1960$) minimise difference between model prices and CORI index values.

Risk-neutral survival curve is the best-estimate survival curve multiplied by an adjustment term $e^{C^i(t,T)}$.

$C^i(t, T)$ is a function of the market prices of longevity risk.

Assume similar cohorts born from 1941 to 1950 share the same $\lambda^{1}_{\mu,3}$ while cohorts born from 1951 to 1960 also share the same $\lambda^{2}_{\mu,3}$.

Calibrate market price of longevity risk by minimizing the error term

$$\hat{\Lambda} = \underset{\Lambda}{\operatorname{argmin}} \sqrt{\sum_{i=1941}^{1960} \left(\hat{I}^{i}_{x}(0) - I^{CoRI,i}_{x}(0) \right)^{2}}. \tag{25}$$

Implied Price of Longevity Risk

Steps used to solve for $\lambda_{\mu,1}$, $\lambda_{\mu,2}$, $\lambda_{\mu,3}^1$ and $\lambda_{\mu,3}^2$:

- Use yield rate at the end of March 2015 with maturity 1-, 2-, ... 60-year using the AFNS model and then calculate the corresponding discount bond prices;
- Simulate best-estimate survival curves for the 20 cohorts born from 1941 to 1960;
- Adjust the best-estimate survival curves using $\lambda_{\mu,1}$, $\lambda_{\mu,2}$, $\lambda_{\mu,3}^1$ and $\lambda_{\mu,3}^2$ to compute the model index levels for the 20 cohorts;
- Find the estimated $\hat{\lambda}_{\mu,1}$, $\hat{\lambda}_{\mu,2}$, $\hat{\lambda}_{\mu,3}^1$ and $\hat{\lambda}_{\mu,3}^2$ for which the model index level closely matches the CoRI index level.

Implied Price of Longevity Risk

The calibrated risk premiums are given in Table 6.

$\hat{\lambda}_{\mu,1}$	$\hat{\lambda}_{\mu,2}$	$\hat{\lambda}^1_{\mu,3}$	$\hat{\lambda}^2_{\mu,3}$
0.3601	0.0892	0.1099	0.0973

Table 6: Calibrated market price of longevity risk.

Positive prices of risk

Similar price of risk across cohort groups.

Implied Price of Longevity Risk

Table 7 shows model risk-neutral index levels and the values of CoRI indexes published by BlackRock on 31 March 2015.

Cohort	Age	Name	Index level	Risk-neutral index level	Difference
1941	74	CoRI Index 2005	15.26	15.96	0.70
1942	73	CoRI Index 2006	15.94	16.41	0.47
1943	72	CoRI Index 2007	16.61	16.89	0.28
1944	71	CoRI Index 2008	17.28	17.40	0.12
1945	70	CoRI Index 2009	17.95	17.93	-0.02
1946	69	CoRI Index 2010	18.60	18.48	-0.12
1947	68	CoRI Index 2011	19.26	19.05	-0.21
1948	67	CoRI Index 2012	19.93	19.64	-0.29
1949	66	CoRI Index 2013	20.59	20.24	-0.35
1950	65	CoRI Index 2014	21.25	20.85	-0.40
1951	64	CoRI Index 2015	22.19	21.03	-1.16
1952	63	CoRI Index 2016	21.50	20.66	-0.84
1953	62	CoRI Index 2017	20.93	20.29	-0.64
1954	61	CoRI Index 2018	20.35	19.93	-0.42
1955	60	CoRI Index 2019	19.73	19.57	-0.16
1956	59	CoRI Index 2020	19.11	19.21	0.10
1957	58	CoRI Index 2021	18.52	18.85	0.33
1958	57	CoRI Index 2022	17.98	18.50	0.52
1959	56	CoRI Index 2023	17.50	18.13	0.63
1960	55	CoRI Index 2024	16.93	17.77	0.84

*The CoRI Index data is obtained from BlackRock on 31 March 2015.

Table 7: CoRI index level and the risk-neutral index level at the market prices of longevity risk given in Table 6.

Implied Price of Longevity Risk - Sensitivity

		Scenarios						
		1	2	3	4	5	6	
$\hat{\lambda}_{\mu,1}$	0.3601	0.3701	0.3501	0.3601	0.3601	0.3601	0.3601	
$\hat{\lambda}_{\mu,2}$	0.0892	0.0892	0.0892	0.0992	0.0792	0.0892	0.0892	
$\hat{\lambda}^1_{\mu,3}$	0.1099	0.1099	0.1099	0.1099	0.1099	0.1199	0.0999	
$\hat{\lambda}^2_{\mu,3}$	0.0973	0.0973	0.0973	0.0973	0.0973	0.1073	0.0873	
Cohort	Risk-neutral index level							CoRI index level
1941	15.96	16.96	15.17	16.13	15.79	16.26	15.67	15.26
1942	16.41	17.36	15.67	16.57	16.25	16.71	16.12	15.94
1943	16.89	17.79	16.18	17.05	16.74	17.20	16.59	16.61
1944	17.40	18.25	16.73	17.55	17.25	17.71	17.10	17.28
1945	17.93	18.74	17.29	18.08	17.79	18.25	17.62	17.95
1946	18.48	19.25	17.87	18.62	18.35	18.81	18.17	18.60
1947	19.05	19.79	18.47	19.19	18.92	19.38	18.73	19.26
1948	19.64	20.34	19.09	19.77	19.51	19.98	19.31	19.93
1949	20.24	20.90	19.71	20.36	20.12	20.58	19.91	20.59
1950	20.85	21.48	20.35	20.97	20.73	21.20	20.51	21.25
1951	21.03	21.59	20.59	21.15	20.93	21.38	20.70	22.19
1952	20.66	21.19	20.24	20.76	20.55	21.00	20.32	21.50
1953	20.29	20.79	19.89	20.39	20.19	20.64	19.95	20.93
1954	19.93	20.40	19.55	20.02	19.83	20.28	19.58	20.35
1955	19.57	20.01	19.21	19.66	19.47	19.92	19.22	19.73
1956	19.21	19.63	18.88	19.30	19.12	19.57	18.86	19.11
1957	18.85	19.25	18.54	18.94	18.77	19.21	18.51	18.52
1958	18.50	18.87	18.20	18.58	18.42	18.85	18.15	17.98
1959	18.13	18.48	17.86	18.21	18.06	18.49	17.79	17.50
1960	17.77	18.10	17.51	17.84	17.70	18.12	17.43	16.93
Total	376.78	389.17	367.01	379.16	374.48	383.54	370.24	377.41

Implied Price of Longevity Risk

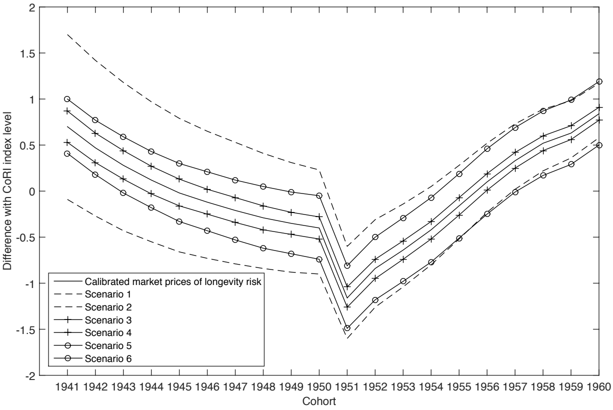

Figure 6: Differences between the CoRI index level and the risk-neutral index level, at calibrated market prices of longevity risk and market prices of longevity risk specified in Scenario 1 to 6.

Longevity Bond Options

Value of a call option at time t as $\text{Call}(r, \mu, t, T_C, T)$, where T_C is the exercise date of the option and T is the maturity of the underlying longevity bond.

Underlying asset for the longevity bond option is a longevity zero-coupon bond maturing after the option expires ($T > T_C$).

The payoff function for the T_C-maturity call option is

$$\text{Call}(r, \mu, T_C, T_C, T) = \left(\bar{P}_x^i(T_C, T) - K\right)^+, \tag{26}$$

where K is the strike price of the option. The price of the longevity bond option at any time t prior to maturity is

$$\text{Call}(r, \mu, t, T_C, T) = E^Q\left[\mathrm{e}^{-\int_t^{T_C}\left(r(s)+\mu^i(x,s)\right)ds}\left(\bar{P}_x^i(T_C, T) - K\right)^+ |\mathcal{F}(t)\right], \tag{27}$$

where $E^Q[\cdot]$ denotes the expectation under the risk-neutral measure.

Longevity Bond Options

Need to eliminate the stochastic discount factor inside the conditional expectation in Equation (27) - see Jamshidian (1989) and Geman et al. (1995).

Use a measure change, from the risk-neutral measure to the T_C-forward measure, linked by the Radon-Nikodym derivative,

$$\frac{dQ^{T_C}}{dQ} = \frac{\exp\{-\int_0^{T_C}(r(s)+\mu^i(x,s))ds\}}{\bar{P}^i_x(0,T_C)}. \tag{28}$$

Price of the longevity bond option at time t is

$$\text{Call}(r,\mu,t,T_C,T) = \bar{P}^i_x(t,T_C)E^{T_C}\left[\left(\bar{P}^i_x(T_C,T)-K\right)^+|\mathcal{F}(t)\right], \tag{29}$$

where $E^{T_C}[\cdot]$ denotes the expectation under the T_C-forward measure.

Longevity Bond Options

Price of a European call option with maturity T and strike K, on a longevity zero-coupon bond with maturity T_C is

$$\begin{aligned}\text{Call}(r,\mu,t,T_C,T) &= \bar{P}_x^i(t,T_C)E^{T_C}\left[\left(\bar{P}_x^i(T_C,T)-K\right)^+|\mathcal{F}(t)\right] \\ &= \bar{P}_x^i(t,T_C)\left[\mathrm{e}^{M_p+\frac{1}{2}V_p^2}\Phi\left(\frac{M_p-\ln K+V_p^2}{V_p}\right)-K\Phi\left(\frac{M_p-\ln K}{V_p}\right)\right], \end{aligned} \tag{30}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

$$\begin{aligned}M_p &= V(T_C,T)+A^i(T_C,T)+C^i(T_C,T)-(T-T_C)E^{T_C}\left[L(T_C)|\mathcal{F}(t)\right] \\ &\quad-\left[\frac{1-e^{-\lambda(T-T_C)}}{\lambda}\right]E^{T_C}\left[S(T_C)|\mathcal{F}(t)\right]-\left[\frac{1-e^{-\lambda(T-T_C)}}{\lambda}-e^{-\lambda(T-T_C)}(T-T_C)\right]E^{T_C}\left[C(T_C)|\mathcal{F}(t)\right] \\ &\quad+B_1(T_C,T)E^{T_C}\left[X_1(T_C)|\mathcal{F}(t)\right]+B_2(T_C,T)E^{T_C}\left[X_2(T_C)|\mathcal{F}(t)\right]+B_3^i(T_C,T)E^{T_C}\left[Z^i(T_C)|\mathcal{F}(t)\right], \end{aligned} \tag{31}$$

Longevity Bond Options

$$
\begin{aligned}
(V_p)^2 &= s_1^2(T - T_C)^2(T_C - t) \\
&+ \left[\frac{1 - e^{-\lambda(T-T_C)}}{\lambda}\right]^2 \left[\frac{s_2^2}{2\lambda}\left(1 - e^{-2\lambda(T_C-t)}\right) + \lambda^2 s_3^2 \int_t^{T_C} (T_C - v)^2 \mathrm{e}^{-2\lambda(T_C-v)} dv\right] \\
&+ \frac{s_3^2}{2\lambda}\left[\frac{1 - e^{-\lambda(T-T_C)}}{\lambda} - e^{-\lambda(T-T_C)}(T - T_C)\right]^2 \left(1 - e^{-2\lambda(T_C-t)}\right) \\
&+ \frac{\sigma_1^2}{2\phi_1^3}\left(1 - e^{-\phi_1(T-T_C)}\right)^2 \left(1 - e^{-2\phi_1(T_C-t)}\right) + \frac{\sigma_2^2}{2\phi_2^3}\left(1 - e^{-\phi_2(T-T_C)}\right)^2 \left(1 - e^{-2\phi_2(T_C-t)}\right) \\
&+ \frac{(\sigma_3^i)^2}{2(\phi_3^i)^3}\left(1 - e^{-\phi_3^i(T-T_C)}\right)^2 \left(1 - e^{-2\phi_3^i(T_C-t)}\right) \qquad (32)
\end{aligned}
$$

with

$$
\begin{aligned}
\int_t^{T_C} (T_C - v)^2 \mathrm{e}^{-2\lambda(T_C-v)} dv = &-\frac{1}{2\lambda}(T_C - t)^2 \mathrm{e}^{-2\lambda(T_C-t)} - \frac{1}{2\lambda^2}(T_C - t)\mathrm{e}^{-2\lambda(T_C-t)} \\
&+ \frac{1}{4\lambda^3}\left[1 - \mathrm{e}^{-2\lambda(T_C-t)}\right].
\end{aligned}
$$

Longevity Bond Options

The price of a European call option with maturity T and strike K, written on the longevity zero-coupon bond with maturity T_C

$$\text{Call}(r, \mu, t, T_C, T) = \bar{P}_x^i(t, T)\Phi\left(\frac{\ln\frac{\bar{P}_x^i(t,T)}{K\bar{P}_x^i(t,T_C)}}{V_p} + \frac{1}{2}V_p\right) - \bar{P}_x^i(t, T_C)K\Phi\left(\frac{\ln\frac{\bar{P}_x^i(t,T)}{K\bar{P}_x^i(t,T_C)}}{V_p} - \frac{1}{2}V_p\right). \tag{33}$$

Noting that $\frac{\bar{P}_x^i(t,T)}{\bar{P}_x^i(t,T_C)}$ is a martingale under Q^{T_C}, so that

$$\frac{\bar{P}_x^i(t, T)}{\bar{P}_x^i(t, T_C)} = E^{T_C}\left[\bar{P}_x^i(T_C, T)|\mathcal{F}(t)\right] = \mathrm{e}^{M_p + \frac{1}{2}V_p^2}. \tag{34}$$

Longevity Bond Options

Compare call option prices on zero coupon bonds with those on zero coupon longevity bonds

Determine prices of call options on longevity zero-coupon bonds for the cohort with starting age 55 in 2015,

- with option maturity 1-, 2-, 5- and 10-year and

- bond maturity 10-, 15-, 20- and 25-year,

- at parameter values given in Table 1 and 5.

Results shown for at-the-money (ATM) options, with strike equal to market price of underlying bond, in-the-money (ITM), with strike equal to 95% of ATM strike, and out-of-the-money (OTM), with strike equal to 105% of ATM strike, options.

Bond Options

Bond maturity	Zero-coupon bond	Option maturity 1	2	5	10
ATM					
10	0.9859	0.0208	0.0255	0.0258	-
15	0.9596	0.0319	0.0412	0.0511	0.0437
20	0.9210	0.0417	0.0553	0.0739	0.0767
25	0.8678	0.0498	0.0670	0.0930	0.1050
ITM					
10	-	0.0531	0.0556	0.0563	-
15	-	0.0604	0.0680	0.0771	0.0740
20	-	0.0675	0.0796	0.0972	0.1014
25	-	0.0731	0.0890	0.1139	0.1263
OTM					
10	-	0.0054	0.0090	0.0090	-
15	-	0.0143	0.0229	0.0320	0.0229
20	-	0.0238	0.0369	0.0551	0.0567
25	-	0.0324	0.0494	0.0753	0.0867

Table 8: Prices of a set of call options on real-rate zero-coupon bonds in 2015, without mortality component.

Longevity Bond Options

		Option maturity			
Bond maturity	Zero-coupon longevity bond	1	2	5	10
ATM					
10	0.9203	0.0303	0.0394	0.0488	-
15	0.8397	0.0396	0.0530	0.0717	0.0830
20	0.7197	0.0435	0.0589	0.0829	0.0993
25	0.5582	0.0407	0.0556	0.0800	0.0986
ITM					
10	-	0.0593	0.0675	0.0797	-
15	-	0.0639	0.0763	0.0954	0.1132
20	-	0.0632	0.0778	0.1015	0.1201
25	-	0.0555	0.0697	0.0935	0.1129
OTM					
10	-	0.0127	0.0204	0.0267	-
15	-	0.0226	0.0352	0.0524	0.0578
20	-	0.0285	0.0436	0.0670	0.0811
25	-	0.0291	0.0438	0.0681	0.0859

Table 9: Prices of a set of call options on longevity zero-coupon bonds for the cohort with starting age 55 in 2015, at the market price of longevity risk given in Table 6.

Longevity Bond Options

Volatility: stochastic mortality adds to interest rate volatility resulting in higher zero coupon longevity bond call option prices compared to zero coupon bond call option prices.

Interest rates: mortality rates equivalent to an effective increase in the interest rate - higher interest rate means higher zero coupon longevity bond call option prices (larger increase for in-the-money, smaller for out-of-the-money)

Strike Price: zero coupon longevity bonds have lower values, due to mortality, compared to zero coupon bonds - produces lower strikes and lower call option prices for zero coupon longevity bond call option prices - larger effect for longer bond maturity.

Zero coupon longevity bond call option prices are slightly hump shaped in bond maturity, increasing at first then decreasing, whereas zero coupon bond call option prices are increasing.

Difference between zero coupon bond call options and zero coupon longevity bond call options reduces with option maturity.

Conclusions and Summary

Estimated a continuous time affine mortality model with cohort effects using US historical data - allows close form for survival curve and incorporation of market prices of risk.

Derived risk neutral model for mortality curve and combined with interest rate term structure model for real rates of interest to replicate Blackrock CORI indexes.

Calibrated market prices of mortality risk for age-period and cohort using Blackrock CORI - shows little variation by cohort factor.

But, market price of risk for second age-period factor has most sensitivity in index values.

Derived call option pricing formula on longevity zero coupon bonds and compared with options on zero coupon bonds - mortality significantly changes the option prices with both bond and option maturity.

Part of an on-going research program.

References

Christensen, J. H., Diebold, F. X., and Rudebusch, G. D. (2011). The affine arbitrage-free class of nelson–siegel term structure models. *Journal of Econometrics*, 164(1):4–20.

Dai, Q. and Singleton, K. J. (2000). Specification analysis of affine term structure models. *The Journal of Finance*, 55(5):1943–1978.

Diebold, F. X. and Li, C. (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics*, 130(2):337–364.

Duffee, G. R. (2002). Term premia and interest rate forecasts in affine models. *The Journal of Finance*, 57(1):405–443.

Geman, H., El Karoui, N., and Rochet, J.-C. (1995). Changes of numeraire, changes of probability measure and option pricing. *Journal of Applied Probability*, pages 443–458.

Jamshidian, F. (1989). An exact bond option formula. *The Journal of Finance*, 44(1):205–209.

Nelson, C. R. and Siegel, A. F. (1987). Parsimonious modeling of yield curves. *Journal of Business*, pages 473–489.

Schönbucher, P. J. (2003). *Credit derivatives pricing models: models, pricing and implementation*. John Wiley & Sons.

Xu, Y., Sherris, M., and Ziveyi, J. (2015). The application of affine processes in multi-cohort mortality models. Working paper, UNSW Business School

Appendix - Kalman Filter

The measurement equation is

$$y_t = -BY_t - A + \varepsilon_t, \qquad \varepsilon_t \sim N(0, H), \tag{35}$$

where A and B are given by (16) and (23), H is a diagonal matrix with elements $\sigma^2_\varepsilon(\tau_i)$. The state transition equation is

$$Y_t = a + bY_{t-1} + \eta_t, \qquad \eta_t \sim N(0, Q), \tag{36}$$

where a, b and Q are given by (18) and (24).

Kalman Filter

Denote the filtered values of the state variables and their corresponding covariance matrix by $Y_{t|t}$ and $S_{t|t}$, and the unknown parameters by θ.

In the forecasting step, forecast unknown values of state variables conditioning on the information at time $t-1$ such that

$$Y_{t|t-1} = a + bY_{t-1|t-1}, \tag{37}$$

$$S_{t|t-1} = b' S_{t-1|t-1} b + Q_t(\theta). \tag{38}$$

Kalman Filter

In the next step use the information at time t to update forecasts

$$Y_{t|t} = Y_{t|t-1} - S_{t|t-1} B(\theta) F_{t|t-1}^{-1} v_{t|t-1}, \tag{39}$$

$$S_{t|t} = S_{t|t-1} - S_{t|t-1} B(\theta) F_{t|t-1}^{-1} B(\theta)' S_{t|t-1}, \tag{40}$$

where

$$\begin{aligned} v_{t|t-1} &= y_t + A(\theta) + B(\theta) X_{t|t-1}, \\ F_{t|t-1} &= B(\theta)' S_{t|t-1} B(\theta) + H. \end{aligned}$$

Kalman Filter

Every iteration will yield a value for the log-likelihood function

$$\log l(y_1, ..., y_T; \theta) = \sum_{t=1}^{T} \left(-\frac{N}{2}\log(2\pi) - \frac{1}{2}\log(F_{t|t-1}) - \frac{1}{2} v'_{t|t-1} F^{-1}_{t|t-1} v_{t|t-1} \right) \tag{41}$$

where N is the number of observed time series.

The estimated parameter set $\hat{\theta}$ maximizes the log-likelihood function. Numerical procedure used to determine maximum.