Valuation of American Options in Nonlinear Market Models

Edward Kim, Tianyang Nie, Marek Rutkowski

School of Mathematics and Statistics, University of Sydney

School of Mathematics, Shandong University

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Standing assumptions

- We deal with a general market model with nonlinear trading.
- Trading conditions are identical for the issuer and the holder of an American contract.
- Their respective initial endowments are arbitrary real numbers x_1 and x_2.
- Let $\mathcal{M} = (\mathcal{B}, \mathcal{S}, \Psi)$ be a derivatives pricing model where:
    - the d-dimensional process $\mathcal{S} = (S^1, S^2, \ldots, S^d)$ represents prices of traded risky asset,
    - the $2(d+1)$-dimensional process $\mathcal{B} = (B^l, B^b, B^1, \ldots, B^{2d})$ represents cash/funding accounts,
    - Φ is the class of all *admissible* trading strategies for all traders.
- The nonlinear market $\mathcal{M} = (\mathcal{B}, \mathcal{S}, \Psi)$ is assumed to be arbitrage-free in a suitable sense (see, for instance, EKQ (1997), BR (2015) or BCR (2016)).
- We focus on unilateral valuations of an American contract $\mathrm{ACC}\,(A, X^h)$ by its issuer and holder at time $t = 0$.
- Game options can ba analyzed using the same mathematical tools.

American contingent claims

Definition (American contingent claim)

- An *American contingent claim* $\mathrm{ACC}\,(A, X^h)$ is a contract between the issuer and the holder who has the right to exercise the contract by selecting a $\mathbb{G}$-stopping time $\tau \in \mathcal{T}_{[0,T]} = \mathcal{T}$.
- Then the issuer receives the amount X^h_τ or, equivalently, he pays to the holder the amount of $-X^h_\tau$ at time τ.
- The process A represents the *cumulative cash flows* from time 0 till $\tau \wedge T$.

- For the American put option on the stock S, the payoff to the issuer of the option equals $X^h_\tau = -(K - S_\tau)^+$ if the option is exercised at time τ by its holder.
- We consider an extended market model $\widetilde{\mathcal{M}}^p$ in which an $\mathrm{ACC}\,(A, X^h)$ is traded at some (yet unknown) initial price p at time 0.
- The goal is to analyze unilateral fair pricing for the issuer and the holder in a general nonlinear framework (later given by unilateral nonlinear evaluations $\mathcal{E}^i$ and $\mathcal{E}^h$).

Monotonicity of the wealth process

- We denote by $\Psi(x, A)$ the class of all admissible trading strategies from Ψ with the initial value $x \in \mathbb{R}$, so that $\Psi = \cup_{x \in \mathbb{R}} \Psi(x, A)$.
- For any initial endowment $x \in \mathbb{R}$, any price $p \in \mathbb{R}$, and any trading strategy $\varphi \in \Psi(x + p, A)$, we denote by $V(x + p, \varphi, A)$ the *wealth process* of φ.
- For brevity, we sometimes write $V(y, \varphi)$ instead of $V(y, \varphi, A)$ when A is fixed.
- For any $x \in \mathbb{R}$ at time 0, the *benchmark wealth* $V^b(x)$ is defined by

$$V^b(x) := xB^l \mathbb{1}_{\{x \geq 0\}} + xB^b \mathbb{1}_{\{x < 0\}}.$$

- The benchmark wealth $V^b(x_1)$ and $V^b(x_2)$ of each party is used to introduce indifference pricing and it can be defined in a more general way.

Assumption (FM = Forward Monotonicity)

For every $p \in \mathbb{R}$, any strategy $\varphi \in \Psi(x_1 + p, A)$ and any $q > p$, there exists a strategy $\widehat{\varphi} \in \Psi(x_1 + q, A)$ such that $V_t(x_1 + q, \widehat{\varphi}, A) \geq V_t(x_1 + p, \varphi, A)$ for every $t \in [0, T]$.

Issuer's Acceptable Price: Definition

Issuer's conditions

Notation

We say that $(p, \varphi, \tau) \in \mathbb{R} \times \Psi(x_1 + p, A) \times \mathcal{T}$ satisfy

$$
\begin{aligned}
\text{(AO)} \quad &\Longleftrightarrow \quad V_\tau(x_1 + p, \varphi, A) + X^h_\tau \geq V^b_\tau(x_1) \\
&\qquad \text{and } \mathbb{P}\big(V_\tau(x_1 + p, \varphi, A) + X^h_\tau > V^b_\tau(x_1)\big) > 0, \\
\text{(SH)} \quad &\Longleftrightarrow \quad V_\tau(x_1 + p, \varphi, A) + X^h_\tau \geq V^b_\tau(x_1), \\
\text{(BG}_\varepsilon\text{)} \quad &\Longleftrightarrow \quad V_\tau(x_1 + p, \varphi, A) + X^h_\tau \leq V^b_\tau(x_1) + \varepsilon, \\
\text{(BE)} \quad &\Longleftrightarrow \quad V_\tau(x_1 + p, \varphi, A) + X^h_\tau = V^b_\tau(x_1), \\
\text{(NA)} \quad &\Longleftrightarrow \quad \text{either } V_\tau(x_1 + p, \varphi, A) + X^h_\tau = V^b_\tau(x_1) \\
&\qquad \text{or } \mathbb{P}\big(V_\tau(x_1 + p, \varphi, A) + X^h_\tau < V^b_\tau(x_1)\big) > 0,
\end{aligned}
$$

Issuer's arbitrage opportunity

Definition (Arbitrage opportunity)

We say that a pair $(p, \varphi) \in \mathbb{R} \times \Psi(x_1 + p)$ *satisfies* (AO) if, for every $\tau \in \mathcal{T}$, the triplet (p, φ, τ) satisfies (AO). We then say that (p, φ) is an *issuer's arbitrage opportunity* in the extended model $\widetilde{\mathcal{M}}^p$ and we write $(p, \varphi) \in$ (AO).

Definition (Superhedging)

We say that a pair $(p, \varphi) \in \mathbb{R} \times \Psi(x_1 + p)$ *satisfies* (SH) if for every $t \in [0, T]$ we have that $V_t(x_1 + p, \varphi) + X_t^h \geq V_t^b(x_1)$. We write $(p, \varphi) \in$ (SH).

Definition (Negligible gains)

We say that $p \in \mathbb{R}$ is an *issuer's superhedging cost with negligible gains* if for every $\varphi \in \Psi(x_1 + p)$ such that $(p, \varphi) \in$ (SH) and for every $\varepsilon > 0$ there exists $\tau^\varepsilon \in \mathcal{T}$ such that $V_{\tau^\varepsilon}(x_1 + p, \varphi) + X_{\tau^\varepsilon}^h \leq V_{\tau^\varepsilon}^b(x_1) + \varepsilon$.

Issuer's fair price

Definition (Fair price)

- A number $p^{f,i} = p^{f,i}(x_1)$ is an *issuer's fair price* if no issuer's arbitrage opportunity (p, φ) arises in $\widetilde{\mathcal{M}}^p$ when $p = p^{f,i}$

$$\mathcal{H}^{f,i}(x_1) := \big\{p \in \mathbb{R} : \forall \varphi \in \Psi(x_1 + p)\, \exists \tau \in \mathcal{T} \text{ s.t. } (p, \varphi, \tau) \in \text{(NA)}\big\}$$

  and the upper bound for issuer's fair prices is given by

$$\overline{p}^{f,i}(x_1) := \sup \big\{p \in \mathbb{R} : p \text{ is an issuer's fair price}\big\}.$$

- If

$$\overline{p}^{f,i}(x_1) = \max \mathcal{H}^{f,i}(x_1)$$

  then $\overline{p}^{f,i}(x_1)$ is denoted as $\widehat{p}^{f,i}(x_1)$ and called the *issuer's maximum fair price*.

Issuer's strict superhedging costs

Definition (Strict superhedging cost)

- The lower bound for *issuer's strict superhedging costs* is given by the equality $\underline{p}^{a,i}(x_1) := \inf \mathcal{H}^{a,i}(x_1)$ where

$$\mathcal{H}^{a,i}(x_1) := \big\{ p \in \mathbb{R} : \exists\, \varphi \in \Psi(x_1 + p) \text{ s.t. } (p, \varphi) \in (\mathrm{AO}) \big\}.$$

- If $\underline{p}^{a,i}(x_1) = \min \mathcal{H}^{a,i}(x_1)$, then we denote it as $\widehat{p}^{a,i}(x_1)$ and it is called the *issuer's minimum strict superhedging cost*.

Definition (Superhedging cost)

- The lower bound for *issuer's superhedging costs* is given by the equality $\underline{p}^{s,i}(x_1) := \inf \mathcal{H}^{s,i}(x_1)$ where

$$\mathcal{H}^{s,i}(x_1) := \big\{ p \in \mathbb{R} : \exists\, \varphi \in \Psi(x_1 + p) \text{ s.t. } (p, \varphi) \in (\mathrm{SH}) \big\}.$$

- If $\underline{p}^{s,i}(x_1) = \min \mathcal{H}^{s,i}(x_1)$, then $\underline{p}^{s,i}(x_1)$ is denoted as $\widehat{p}^{s,i}(x_1)$ and it is called the *issuer's minimum superhedging cost*.

Break-even time

Assumption (SFM = Strict Forward Monotonicity)

For every $p \in \mathbb{R}$, any strategy $\varphi \in \Psi(x_1 + p)$ and any $q > p$, there exists $\widehat{\varphi} \in \Psi(x_1 + q)$ such that $V_t(x_1 + q, \widehat{\varphi}) > V_t(x_1 + p, \varphi)$ for all $t \in [0, T]$.

Lemma

Under Assumption (SFM), we have $\overline{p}^{f,i}(x_1) = \underline{p}^{s,i}(x_1) = \underline{p}^{a,i}(x_1)$.

Definition (Break-even time)

If condition (BE) is satisfied by (p, φ, τ), then τ is called an *issuer's break-even time* for the pair $(p, \varphi) \in \mathbb{R} \times \Psi(x_1 + p)$.

- Any issuer's break-even time is one of the exercise times available to the holder. However, it is unlikely that it will be a 'rational' exercise time for the holder.
- It may not be advantageous for the holder to exercise at a stopping time that causes the issuer to break even or prohibits the issuer's arbitrage opportunities.

Issuer's replication costs

Definition (Replication cost)

- The lower bound for *issuer's replication costs* equals $\underline{p}^{r,i}(x_1) := \inf \mathcal{H}^{r,i}(x_1)$ where

$$\mathcal{H}^{r,i}(x_1) = \big\{p \in \mathbb{R} : \exists\, (\varphi, \tau) \in \Psi(x_1 + p) \times \mathcal{T} \text{ s.t. } (p, \varphi) \in \text{(SH)} \\ \text{and } (p, \varphi, \tau) \in \text{(BE)}\big\}.$$

- If

$$\underline{p}^{r,i}(x_1) = \min \mathcal{H}^{r,i}(x_1)$$

then $\underline{p}^{r,i}(x_1)$ is denoted as $\widehat{p}^{r,i}(x_1)$ and called the *issuer's minimum replication cost*.

- It is worth noting that the issuer's minimum replication cost is not necessarily an issuer's fair price for $\mathrm{ACC}\,(A, X^h)$.

Issuer's fair replication price

Definition (Fair replication cost)

- The lower bound for *issuer's fair replication costs* is given by the equality $\underline{p}^{r,f,i}(x_1) := \inf \mathcal{H}^{f,r,i}(x_1)$ where

$$\mathcal{H}^{f,r,i}(x_1) = \big\{p \in \mathbb{R} : \exists\,(\varphi,\tau) \in \Psi(x_1+p) \times \mathcal{T} \text{ s.t. } (p,\varphi) \in \text{(SH)} \\ \text{and } (p,\varphi,\tau) \in \text{(BE)};\ \forall\,\psi \in \Psi(x_1+p)\,\exists\,\tau' \in \mathcal{T} \text{ s.t. } (p,\psi,\tau') \in \text{(NA)}\big\}.$$

- If

$$\underline{p}^{f,r,i}(x_1) = \min \mathcal{H}^{f,r,i}(x_1)$$

then $\underline{p}^{f,r,i}(x_1)$ is denoted as $\widehat{p}^{f,r,i}(x_1)$ we call it the *issuer's fair replication price*.

Definition (Acceptable price)

If the set $\mathcal{H}^{f,r,i}(x_1)$ is a singleton, then its unique element $p^i(x_1)$ is called the *issuer's acceptable price*.

General properties of issuer's costs

- Recall the notation:
  - $\widehat{p}^{f,i}$ – the maximum of issuer's fair prices,
  - $\breve{p}^{s,i}$ – the minimum of issuer's superhedging costs,
  - $\breve{p}^{r,i}$ – the minimum of issuer's replication costs,
  - $\breve{p}^{f,r,i}$ – the minimum of issuer's fair replication prices.

Proposition

Assume that the property (SFM) holds. Then:
(i) if $\mathcal{H}^{f,r,i}(x_1) \neq \emptyset$, then it is a singleton and the issuer's acceptable price $p^i = p^i(x_1)$ satisfies

$$-\infty < \widehat{p}^{f,i} = p^i = \breve{p}^{f,r,i} = \breve{p}^{r,i} = \breve{p}^{s,i} < +\infty$$

(ii) if $\mathcal{H}^{r,i}(x_1) \neq \emptyset$, then $\overline{p}^{f,i} = \underline{p}^{s,i} \leq \underline{p}^{r,i} < \infty$,
(iii) if $\mathcal{H}^{r,i}(x_1) = \emptyset$, then $\overline{p}^{f,i} = \underline{p}^{s,i} \leq \underline{p}^{r,i} = \infty$.

Holder's Acceptable Price: Definition

Holder's conditions

Notation

We say that $(p, \psi, \tau) \in \mathbb{R} \times \Psi(x_2 - p, -A) \times \mathcal{T}$ satisfy

$$\begin{aligned}
\text{(AO')} \quad &\Longleftrightarrow \quad V_\tau(x_2 - p, \psi, -A) - X^h_\tau \geq V^b_\tau(x_2) \\
&\qquad\qquad \text{and } \mathbb{P}\big(V_\tau(x_2 - p, \psi, -A) - X^h_\tau > V^b_\tau(x_2)\big) > 0, \\
\text{(SH')} \quad &\Longleftrightarrow \quad V_\tau(x_2 - p, \psi, -A) - X^h_\tau \geq V^b_\tau(x_2), \\
\text{(BE')} \quad &\Longleftrightarrow \quad V_\tau(x_2 - p, \psi, -A) - X^h_\tau = V^b_\tau(x_2), \\
\text{(NA')} \quad &\Longleftrightarrow \quad \text{either } V_\tau(x_2 - p, \psi, -A) - X^h_\tau = V^b_\tau(x_2) \\
&\qquad\qquad \text{or } \mathbb{P}\big(V_\tau(x_2 - p, \psi, -A) - X^h_\tau < V^b_\tau(x_2)\big) > 0, \\
\text{(SB')} \quad &\Longleftrightarrow \quad \mathbb{P}\big(V_\tau(x_2 - p, \psi, -A) - X^h_\tau < V^b_\tau(x_2)\big) > 0.
\end{aligned}$$

Holder's fair price

Definition (Arbitrage opportunity)

A *holder's arbitrage opportunity in the extended model* $\widetilde{\mathcal{M}}^p$ is an arbitrary triplet $(p, \psi, \tau) \in \mathbb{R} \times \Psi(x_2 - p, -A) \times \mathcal{T}$ satisfying condition (AO').

Definition (Fair price)

- A number $p^{f,h} = p^{f,h}(x_2)$ is a *holder's fair price* if no holder's arbitrage opportunity (p, ψ, τ) arises in $\widetilde{\mathcal{M}}^p$ when $p = p^{f,h}$

$$\mathcal{H}^{f,h}(x_2) := \big\{p \in \mathbb{R} : \forall\,(\psi, \tau) \in \Psi(x_2 - p) \times \mathcal{T}\ (p, \psi, \tau) \in (\text{NA'})\big\}$$

  and the lower bound for the holder's fair prices is given by

$$\underline{p}^{f,h}(x_2) := \inf\big\{p \in \mathbb{R} : p \text{ is a holder's fair price}\big\}.$$

- If $\underline{p}^{f,h}(x_2) = \min\, \mathcal{H}^{f,h}(x_2)$, then $\underline{p}^{f,h}(x_2)$ is denoted as $\widehat{p}^{f,h}(x_2)$ and it is called the *holder's minimum fair price*.

Holder's superhedging costs

Definition (Strict superhedging cost)

- The upper bound for *holder's strict superhedging costs* for $\mathrm{ACC}\,(A, X^h)$ is given by $\overline{p}^{a,h}(x_2) := \sup\, \mathcal{H}^{a,h}(x_2)$ where

$$\mathcal{H}^{a,h}(x_2) := \big\{p \in \mathbb{R} : \exists\,(\psi,\tau) \in \Psi(x_2 - p) \times \mathcal{T} \ \text{ s.t. } (p,\psi,\tau) \in \text{(AO')}\big\}.$$

- If $\overline{p}^{a,h}(x_2) = \max\, \mathcal{H}^{a,h}(x_2)$, then $\overline{p}^{a,h}(x_2)$ is denoted as $\widehat{p}^{a,h}(x_2)$ and called the *holder's maximum strict superhedging cost*.

Definition (Superhedging cost)

- The upper bound for *holder's superhedging costs* is given by the equality $\overline{p}^{s,h}(x_2) := \sup\, \mathcal{H}^{s,h}(x_2)$ where

$$\mathcal{H}^{s,h}(x_2) := \big\{p \in \mathbb{R} : \exists\,(\psi,\tau) \in \Psi(x_2 - p) \times \mathcal{T} \ \text{ s.t. } (p,\psi,\tau) \in \text{(SH')}\big\}.$$

- If $\overline{p}^{s,h}(x_2) = \max\, \mathcal{H}^{s,h}(x_2)$, then $\overline{p}^{s,h}(x_2)$ is denoted as $\widehat{p}^{s,h}(x_2)$ and it is called the *holder's maximum superhedging cost*.

Holder's replication costs

Definition (Replication cost)

- The upper bound for *holder's replication costs* for $\mathrm{ACC}\,(A, X^h)$ is given by

$$\overline{p}^{r,h}(x_2) := \sup\, \mathcal{H}^{r,h}(x_2)$$

  where

$$\mathcal{H}^{r,h}(x_2) = \big\{p \in \mathbb{R} : \exists\,(\psi,\tau) \in \Psi(x_2 - p) \times \mathcal{T} \;\; \text{s.t.} \;\; (p,\psi,\tau) \in \text{(BE')}\big\}.$$

- Equivalently,

$$\overline{p}^{r,h}(x_2) := -\inf\big\{q \in \mathbb{R} : \;\exists\,(\psi,\tau) \in \Psi(x_2 + q) \times \mathcal{T} \\ V_\tau(x_2 + q, \psi) - X^h_\tau = V^b_\tau(x_2)\big\}.$$

- If $\overline{p}^{r,h}(x_2) = \max\, \mathcal{H}^{r,h}(x_2)$, then it is denoted as $\widehat{p}^{r,h}(x_2)$ and called the *holder's maximum replication cost*.

Holder's fair replication costs

Definition (Fair replication cost)

- The upper bound for *holder's fair replication costs* is given by the equality

$$\overline{p}^{f,r,h}(x_2) = \sup \mathcal{H}^{f,r,h}(x_2)$$

where

$$\mathcal{H}^{f,r,h}(x_2) := \big\{ p \in \mathbb{R} : \exists\, (\psi, \tau) \in \Psi(x_2 - p) \times \mathcal{T}\ (p, \varphi, \tau) \in \text{(BE')} \\ \text{and } \forall\, (\varphi, \tau') \in \Psi(x_2 - p) \times \mathcal{T}\, (p, \varphi, \tau') \in \text{(NA')} \big\}.$$

- If $\overline{p}^{f,r,h}(x_2) = \max \mathcal{H}^{f,r,h}(x_2)$, then it is denoted as $\widehat{p}^{f,r,h}(x_2)$ and called the *holder's fair replication price*.

Definition (Acceptable price)

If the set $\mathcal{H}^{f,r,h}(x_2)$ is a singleton, then its unique element $p^h(x_2)$ is called the *acceptable holder's price*.

General properties of holder's costs

- Recall the notation:
  - $\breve{p}^{f,h}$ – the minimum of holder's fair prices,
  - $\widehat{p}^{s,h}$ – the maximum of holder's superhedging costs,
  - $\widehat{p}^{r,h}$ – the maximum of holder's replication costs,
  - $\widehat{p}^{f,r,h}$ – the maximum of holder's fair replication prices.

Proposition

Assume that the property (SFM) holds. Then:
(i) if $\mathcal{H}^{f,r,h}(x_2) \neq \emptyset$, *then it is a singleton, the holder's acceptable price* $p^h = p^h(x_2)$ *satisfies*

$$-\infty < \breve{p}^{f,h} = p^h = \widehat{p}^{f,r,h} = \widehat{p}^{r,h} = \widehat{p}^{s,h} < +\infty$$

(ii) if $\mathcal{H}^{r,h}(x_2) \neq \emptyset$, *then* $-\infty < \overline{p}^{r,h} \leq \underline{p}^{f,h} = \overline{p}^{s,h}$,
(iii) if $\mathcal{H}^{r,h}(x_2) = \emptyset$, *then* $-\infty = \overline{p}^{r,h} \leq \underline{p}^{f,h} = \overline{p}^{s,h}$.

Nonlinear $\mathcal{E}$valuations

Nonlinear evaluation $\mathcal{E}$

Assumption (A.1)

- For every $\sigma, \tau \in \mathcal{T}$, $\sigma \le \tau$, $y_\sigma, y'_\sigma \in \mathcal{L}(\mathcal{G}_\sigma)$ and all trading strategies $\varphi \in \Psi^\sigma(y_\sigma, A)$ and $\varphi' \in \Psi^\sigma(y'_\sigma, A)$ the following implication holds: if $V_\tau(y'_\sigma, \varphi', A) \ge V_\tau(y_\sigma, \varphi, A)$ on some event $E \in \mathcal{G}_\sigma$, then $y'_\sigma \ge y_\sigma$ on E.
- If, in addition, $y'_\sigma = y_\sigma$ on E, then $V_\tau(y'_\sigma, \varphi', A) = V_\tau(y_\sigma, \varphi, A)$ on E.
- For every $\tau \in \mathcal{T}$ and any random variable $\zeta_\tau \in \mathcal{L}(\mathcal{G}_\tau)$ there exists a pair $(p, \varphi) \in \mathbb{R} \times \Psi(x_1 + p, A)$ satisfying $V_\tau(x_1 + p, \varphi, A) = \zeta_\tau$.

Definition (Nonlinear evaluation)

The *nonlinear evaluation* $\mathcal{E} : \mathcal{L}(\mathcal{G}_T) \to \mathcal{L}(\mathcal{G}_T)$ is the mapping such that for all stopping times $\sigma \le \tau$ and every random variable $\zeta_\tau \in \mathcal{L}(\mathcal{G}_\tau)$ we have $\mathcal{E}_{\sigma,\tau}(\zeta_\tau) = y_\sigma$ where $(y_\sigma, \varphi) \in \mathcal{L}(\mathcal{G}_\sigma) \times \Psi^\sigma(y_\sigma, A)$ is such that $V_\tau(y_\sigma, \varphi, A) = \zeta_\tau$.

Nonlinear evaluation $\mathcal{E}$

Assumption (A.2)

The following *concatenation* property of trading strategies holds: for every $\sigma, \rho \in \mathcal{T}$ such that $\sigma \leq \rho$, if $\varphi' \in \Psi^{\sigma}(y_\sigma, A)$ and $\varphi'' \in \Psi^{\rho}(y_\rho, A)$ where $y_\rho = V_\rho(y_\sigma, \varphi', A)$, then the process

$$\varphi = \mathbb{1}_{[\sigma,\rho]}\psi' + \mathbb{1}_{(\rho,T]}\psi''$$

belongs to $\Psi^{\sigma}(y_\sigma, A)$ and

$$V(y_\sigma, \varphi, A) = \mathbb{1}_{[\sigma,\rho]}V(y_\sigma, \varphi', A) + \mathbb{1}_{(\rho,T]}V(y_\rho, \varphi'', A).$$

Lemma

If Assumptions (A.1) and (A.2) are satisfied, then $\mathcal{E}$ is time-consistent, meaning that $\mathcal{E}_{\sigma,\rho}(\mathcal{E}_{\rho,\tau}(\zeta_\tau)) = \mathcal{E}_{\sigma,\tau}(\zeta_\tau)$ for all $\sigma \leq \rho \leq \tau$ and $\zeta_\tau \in \mathcal{L}(\mathcal{G}_\tau)$.

Nonlinear evaluation $\mathcal{E}$

Assumption (A.3)

- Let $\sigma, \sigma', \tau, \tau' \in \mathcal{T}$, $y_\sigma \in \mathcal{G}_\sigma$, $y'_{\sigma'} \in \mathcal{G}_{\sigma'}$. Assume that the following conditions hold on some $E \in \mathcal{G}_\sigma$: $y_\sigma = y'_{\sigma'}$ and $\sigma = \sigma' \leq \tau = \tau'$. If the strategies $\varphi \in \Psi^\sigma(y_\sigma, A)$ and $\varphi' \in \Psi^\sigma(y'_{\sigma'}, A)$ are such that $\varphi = \varphi'$ on $E \times [\sigma, \tau]$, then the equality $V(y_\sigma, \varphi, A) = V(y'_{\sigma'}, \varphi', A)$ holds on $E \times [\sigma, \tau]$.
- Let $\sigma \in \mathcal{T}, y'_\sigma, y''_\sigma \in \mathcal{G}_\sigma$. For every $\varphi' \in \Psi^\sigma(y'_\sigma, A)$ and $\varphi'' \in \Psi^\sigma(y''_\sigma, A)$ the strategy $\varphi = \mathbb{1}_E \varphi' + \mathbb{1}_{E^c} \varphi''$ belongs to $\Psi^\sigma(y_\sigma, A)$ where $y_\sigma = \mathbb{1}_E y'_\sigma + \mathbb{1}_{E^c} y''_\sigma$.

Lemma

- *Let Assumptions (A.1)–(A.3) hold and $\sigma, \tau, \tau' \in \mathcal{T}$ be such that $\sigma \leq \tau \wedge \tau'$. If $\tau = \tau'$ on some $E \in \mathcal{G}_\sigma$ and $\zeta \in \mathcal{L}(\mathcal{G}_\tau), \zeta' \in \mathcal{L}(\mathcal{G}_{\tau'})$ are such that $\zeta = \zeta'$ on E, then $\mathcal{E}_{\sigma,\tau}(\zeta) = \mathcal{E}_{\sigma,\tau'}(\zeta')$ on E.*
- *If, in addition, the equality $\sigma = \sigma'$ holds on E, then $\mathcal{E}_{\sigma,\tau}(\zeta) = \mathcal{E}_{\sigma',\tau'}(\zeta')$ on E.*

Nonlinear evaluation $\mathcal{E}$

Recall that Peng (2004) introduced the concept of the *time-consistent nonlinear evaluation*.

Proposition

If Assumptions (A.1)–(A.3) hold, then the system of operators

$$\mathcal{E} := \left\{ \mathcal{E}_{\sigma,\tau} : \mathcal{L}(\mathcal{G}_\tau) \to \mathcal{L}(\mathcal{G}_\sigma) \,|\, \sigma, \tau \in \mathcal{T}, \, \sigma \leq \tau \right\}$$

has the following properties, for all $\sigma, \tau \in \mathcal{T}$ and $\zeta_\tau, \eta_\tau \in \mathcal{L}(\mathcal{G}_\tau)$:

- $\mathcal{E}_{\sigma,\tau}(\zeta_\tau) \geq \mathcal{E}_{\sigma,\tau}(\eta_\tau)$ if $\zeta_\tau \geq \eta_\tau$,
- $\mathcal{E}_{\tau,\tau}(\zeta_\tau) = \zeta_\tau$,
- $\mathcal{E}_{\sigma,\tau}(\zeta_\tau) = \mathcal{E}_{\sigma,\rho}(\mathcal{E}_{\rho,\tau}(\zeta_\tau))$ for all $\sigma \leq \rho \leq \tau$,
- $\mathbb{1}_E \mathcal{E}_{\sigma,\tau}(\zeta_\tau) = \mathbb{1}_E \mathcal{E}_{\sigma,\tau}(\mathbb{1}_E \zeta_\tau)$ for every $E \in \mathcal{G}_\sigma$

so that $\mathcal{E}$ is the time-consistent nonlinear evaluation.

Nonlinear Optimal Stopping

$\mathcal{E}$-Snell envelope

Definition ($\mathcal{E}$-supermartingale)

A $\mathbb{G}$-optional, làdlàg process Y is called a *strong $\mathcal{E}$-supermartingale* (respectively, a *strong $\mathcal{E}$-martingale*) if $Y_\sigma \geq \mathcal{E}_{\sigma,\tau}(Y_\tau)$ (respectively, $Y_\sigma = \mathcal{E}_{\sigma,\tau}(Y_\tau)$) for all $\sigma, \tau \in \mathcal{T}$ such that $\sigma \leq \tau$.

Let the payoff process ξ upon stopping be an arbitrary $\mathbb{G}$-optional, làdlàg process.

Definition ($\mathcal{E}$-Snell envelope)

Let $\breve{\xi}$ satisfy the following conditions:

- $\breve{\xi}$ is a strong $\mathcal{E}$-supermartingale,
- $\breve{\xi} \geq \xi$,
- for any strong $\mathcal{E}$-supermartingale $Y \geq \xi$ the inequality $Y \geq \breve{\xi}$ holds.

Then we say that $\breve{\xi}$ is the *$\mathcal{E}$-Snell envelope* of ξ and we write $\breve{\xi} = \mathcal{E}\text{-Snell}\,(\xi)$.

To establish the existence of the Snell envelope for ξ, we will also postulate the *continuity* property of trading strategies and thus also of the nonlinear evaluation $\mathcal{E}$.

$\mathcal{E}$-Snell envelope

Assumption (A.4)

- Let $\sigma \leq \tau \leq \tau_n$ where τ_n is a nonincreasing sequence of stopping time such that $\lim_{n\to\infty} \tau_n = \tau$ and let $\zeta \in \mathcal{G}_\tau, \zeta_n \in \mathcal{G}_{\tau_n}$ satisfy $\lim_{n\to\infty} \zeta_n = \zeta$. If y^n_σ and y_σ are such that $V_{\tau_n}(y^n_\sigma, \varphi^n, A) = \zeta_n$ and $V_\tau(y_\sigma, \varphi, A) = \zeta$ for some $\varphi_n \in \Psi^\sigma(y^n_\sigma, A)$ and $\varphi \in \Psi^\sigma(y_\sigma, A)$, then $\lim_{n\to\infty} y^n_\sigma = y_\sigma$.
- Consequently, the convergence $\lim_{n\to\infty} \mathcal{E}_{\sigma,\tau_n}(\zeta_n) = \mathcal{E}_{\sigma,\tau}(\zeta)$ is valid.

Proposition

- *Let Assumptions (A.1)–(A.4) hold. Then for any $\mathbb{G}$-optional, làdlàg process ξ such that $\xi_\tau \in \mathcal{L}(\mathcal{G}_\tau)$ for every $\tau \in \mathcal{T}$, the $\mathcal{E}$-Snell envelope is well-defined and it is a right-upper-semicontinuous $\mathcal{E}$-supermartingale.*
- *Moreover, the equality $\breve{\xi}_\sigma = \xi_\sigma \vee \widehat{\xi}_\sigma$ holds for every $\sigma \in \mathcal{T}$ where $\widehat{\xi}$ is given by*

$$\widehat{\xi}_\sigma = \operatorname*{ess\,sup}_{\tau\in\mathcal{T},\,\tau>\sigma} \mathcal{E}_{\sigma,\tau}(\xi_\tau)$$

where $\mathcal{T}_{(\sigma,T]} = \big\{\tau \in \mathcal{T} \,|\, \tau > \sigma \text{ on the event } \{\tau < T\}\big\}$.

Value of $\mathcal{E}$-optimal stopping problem

Definition ($\mathcal{E}$-optimal stopping)

The *value* of the $\mathcal{E}$-optimal stopping problem is given by

$$\overline{v}_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}_{0,\tau}(\xi_\tau).$$

We say that $\tau^* \in \mathcal{T}$ is a *solution* to the $\mathcal{E}$-optimal stopping problem if

$$\overline{v}_0 = \mathcal{E}_{0,\tau^*}(\xi_{\tau^*}) = \max_{\tau \in \mathcal{T}} \mathcal{E}_{0,\tau}(\xi_\tau).$$

The following lemma is (almost) classical – see Grigorova et al. (2017).

Lemma

- *The equality $\overline{v}_0 = \breve{\xi}_0$ holds where $\breve{\xi} = \mathcal{E}\text{-Snell}\,(\xi)$.*
- *A stopping time τ^* is a solution to the $\mathcal{E}$-optimal stopping problem if and only if $\breve{\xi}_{\tau^*} = \xi_{\tau^*}$ and $\breve{\xi}$ is a strong $\mathcal{E}$-martingale on $[0, \tau^*]$.*

Doob-Meyer-Mertens representation

- Our next goal is to establish the Doob-Meyer-Mertens (DMM) representation for an arbitrary strong $\mathcal{E}$-supermartingale (for the classical case, see Mertens (1972)).
- Peng (1999, 2005), Cohen (2012), Bouchard et al. (2016) and Grigorova et al. (2017) examined the DMM representation for g-supermartingales associated with a BSDE with the generator g.

Assumption (A.5)

Let $U, U' \in \mathcal{U}$ be such that $U' - U$ is increasing. For all $\sigma \in \mathcal{T}$, $y_\sigma \in \mathcal{G}_\sigma$, and every issuer's trading strategy $\varphi \in \Psi^\sigma(y_\sigma, A+U)$ there exists $\varphi' \in \Psi^\sigma(y_\sigma, A+U')$ such that $V_t(y_\sigma, \varphi', A+U') \geq V_t(y_\sigma, \varphi, A+U)$ for every $t \in [\sigma, T]$.

Lemma

Let Assumptions (A.1)–(A.5) be satisfied. If U is a $\mathbb{G}$-optional, decreasing process and $Y_t = \mathcal{E}_{t,T}(Y_T, U)$ for all $t \in [0,T]$, then Y is a strong $\mathcal{E}$-supermartingale.

Doob-Meyer-Mertens representation

It is natural to conjecture that for any strong $\mathcal{E}$-supermartingale there exists a $\mathbb{G}$-optional, làdlàg, decreasing process U such that $Y_t = \mathcal{E}_{t,T}(Y_T, U)$ for all $t \in [0,T]$.

Definition (Mertens conditions)

We say that the processes R^c, R^g and R^d satisfy the *Mertens conditions* if they are increasing processes with $R^c_0 = R^g_0 = R^d_{0-} = 0$, R^c is $\mathbb{G}$-optional and continuous, R^g and R^d are càdlàg, pure jump processes, R^g is $\mathbb{G}$-predictable and R^d is $\mathbb{G}$-optional.

- If (R^c, R^g, R^d) satisfy the Mertens conditions, then we write $(R^c, R^g, R^d) \in \mathcal{R}$.
- We define the làdlàg, decreasing process $R = -R^c - R^g - R^d_-$.

Assumption (A.6)

The processes A and $U \in \mathcal{U}$ are càdlàg and for every $\varphi \in \Psi(y, A+U)$ and all $\tau \in \mathcal{T}$ we have

$$\Delta V_\tau(y, \varphi, A+U) = \Delta V_\tau(y, \varphi^\tau, A^{\tau-} + U^{\tau-}) + \Delta(A_\tau + U_\tau).$$

Doob-Meyer-Mertens representation

Proposition

Let $\bar{\xi}$ satisfy the following assumptions:

- $\bar{\xi}$ is a strong $\mathcal{E}$-supermartingale,
- $\bar{\xi} \geq \xi$,
- the Skorokhod conditions are valid for some $R \in \mathcal{R}$

$$\int_0^T (\bar{\xi}_{t-} - \xi_{t-})\, d\breve{R}^c_t = 0, \quad \Delta \breve{R}^g_t = \Delta \breve{R}^g_t\, \mathbb{1}_{\{\bar{\xi}_{t-} = \xi_{t-}\}}, \quad \Delta \breve{R}^d_t = \Delta \breve{R}^d_t\, \mathbb{1}_{\{\bar{\xi}_t = \xi_t\}}.$$

Then $\bar{\xi}$ satisfies for every $\sigma \in \mathcal{T}$

$$\bar{\xi}_\sigma = \overline{v}_\sigma = \sup_{\tau \in \mathcal{T},\, \tau \geq \sigma} \mathcal{E}_{\sigma,\tau}(\xi_\tau).$$

Proposition (DMM)

Let Assumptions (A.1)–(A.6) hold. If Y is a strong $\mathcal{E}$-supermartingale, then there exists a unique triplet $(U^c, U^g, U^d) \in \mathcal{R}$ such that $Y_t = \mathcal{E}_{t,T}(Y_T, U)$ for all $t \in [0,T]$.

Solution to $\mathcal{E}$-optimal stopping problem

Proposition (DMM for $\breve{\xi}$)

The Snell envelope $\breve{\xi}$ satisfies $\breve{\xi}_t = \mathcal{E}_{t,T}(\xi_T, \breve{R})$ for all $t \in [0,T]$ where $(\breve{R}^c, \breve{R}^g, \breve{R}^d) \in \mathcal{R}$ and

$$\int_0^T (\breve{\xi}_{t-} - \xi_{t-})\, d\breve{R}_t^c = 0, \quad \Delta \breve{R}_t^g = \Delta \breve{R}_t^g\, \mathbb{1}_{\{\breve{\xi}_{t-} = \xi_{t-}\}}, \quad \Delta \breve{R}_t^d = \Delta \breve{R}_t^d\, \mathbb{1}_{\{\breve{\xi}_t = \xi_t\}}.$$

The following proposition completes the solution to the $\mathcal{E}$-optimal stopping problem.

Proposition

If ξ is a right-upper-semicontinuous process and $\breve{R}_{\tau^*} = 0$ where

$$\tau^* := \inf\{t \in [0,T] \,|\, \breve{\xi}_t = \xi_t\}$$

then τ^* is a solution to the $\mathcal{E}$-optimal stopping problem with the payoff process ξ.

Issuer's Acceptable Price: Solution

Backward monotonicity

We consider issuer's valuation and we henceforth denote $X = V^b(x_1) - X^h$.

Assumption (BM = Backward Monotonicity)

For every $x, p, p' \in \mathbb{R}, \varphi \in \Psi(x+p), \varphi' \in \Psi(x+p')$ and an arbitrary $\tau \in \mathcal{T}$, if $V_\tau(x+p', \varphi') \geq V_\tau(x+p, \varphi)$, then $p' \geq p$.

Definition (Operator $\mathcal{E}^i$)

Under assumption (BM), if for some $\tau \in \mathcal{T}$ there exists a pair $(y, \varphi) \in \mathbb{R} \times \Psi(y, A)$ satisfying $V_\tau(y, \varphi) = X_\tau$, then y is unique and it is denoted by $\mathcal{E}^i_{0,\tau}(X)$.

Assumption (Issuer's completeness)

We assume that X is *replicable* for the issuer, in the sense that for a given $x_1 \in \mathbb{R}$ and every $\tau \in \mathcal{T}$ there exists a pair $(p, \varphi) \in \mathbb{R} \times \Psi(x_1 + p)$ satisfying $V_\tau(x_1 + p, \varphi) = X_\tau$. Hence the mapping $\mathcal{E}^i$ is well defined.

Issuer's optimal stopping problem

Definition (Issuer's optimal stopping)

We say that $(v_0^i(x_1), \tau^{*,i}) \in \mathbb{R} \times \mathcal{T}$ is a solution to the *issuer's optimal stopping problem* for $\mathrm{ACC}\,(A, X^h)$ if $v_0^i(x_1) = \mathcal{E}^i_{0,\tau^{*,i}}(X) - x_1$ where $\tau^{*,i} \in \mathcal{T}$ is such that

$$\mathcal{E}^i_{0,\tau^{*,i}}(X) = \max_{\tau \in \mathcal{T}} \mathcal{E}^i_{0,\tau}(X)$$

Assumption (IOS)

The issuer's optimal stopping problem has a solution $(v_0^i(x_1), \tau^{*,i})$. Furthermore, for $p^{*,i} = v_0^i(x_1)$ there exists a strategy $\varphi^* \in \Psi(x_1 + p^{*,i})$ such that $(p^{*,i}, \varphi^*)$ satisfy (SH) and $(p^{*,i}, \varphi^*, \tau^{*,i})$ satisfy (BE). Hence $\mathcal{H}^{r,i}(x_1) \neq \emptyset$.

Lemma

(i) If Assumptions (BM) and (IOS) are satisfied, then $\underline{p}^{s,i}(x_1) \geq v_0^i(x_1)$.
(ii) If Assumptions (SFM), (BM) and (IOS) are met, then $\overline{p}^{f,i}(x_1) \leq v_0^i(x_1)$.

Superhedging cost with negligible gains

Assumption (SBM = Strict Backward Monotonicity)

- For every $x, p, p' \in \mathbb{R}, \varphi \in \Psi(x+p), \varphi' \in \Psi(x+p')$, and an arbitrary $\tau \in \mathcal{T}$, if $V_\tau(x+p', \varphi') \geq V_\tau(x+p, \varphi)$, then $p' \geq p$.
- If, in addition, $V_\tau(x+p', \varphi') \neq V_\tau(x+p, \varphi)$, then $p' > p$.

Definition (Value $v_0^i(x_1)$)

We say that $v_0^i(x_1) \in \mathbb{R}$ is the *value* of the issuer's stopping problem if

$$v_0^i(x_1) = \sup_{\tau \in \mathcal{T}} \mathcal{E}^i_{0,\tau}(X_\tau) - x_1.$$

Proposition

Let Assumptions (SFM) and (SBM) be satisfied. If X is a right-upper-semicontinuous process, then the issuer's minimum superhedging cost for $\mathrm{ACC}\,(X^h, A)$ is equal to the minimal issuer's superhedging cost with negligible gains for $\mathrm{ACC}\,(X^h, A)$.

Issuer's acceptable price

Assumption (OS)

Let $\breve{R}_{\tau^{*,i}} = 0$ where $\tau^{*,i} := \inf\{t \in [0,T] \,|\, \breve{X}_t = X_t\}$ and $\breve{X}$ is the $\mathcal{E}^i$-Snell envelope of X. Note that $\breve{X}_T = X_T$.

Lemma

Let Assumptions (SFM), (SBM) and (OS) hold. If X is right-upper-semicontinuous, then $\tau^{*,i}$ is a solution to the issuer's optimal replication problem.

Theorem (Issuer's valuation)

If X is right-upper-semicontinuous, then the acceptable issuer's price satisfies

$$p^i(x_1) = \widehat{v}^i(x_1) = \widehat{p}^{f,i}(x_1) = \breve{p}^{r,i}(x_1) = \breve{p}^{s,i}(x_1),$$

and the stopping time $\tau^{*,i}$ is an issuer's breakeven time for (p^i, φ^i) where φ^i is given by

$$\mathcal{E}^i_{0,\tau}(X) = x_1 + \big\{p \in \mathbb{R} \,|\, \exists\, \varphi \in \Psi(x_1 + p, A) \colon V_\tau(x_1 + p, \varphi, A) = X_\tau\big\},$$

with $p = p^i(x_1)$.

Holder's Acceptable Price: Solution

Holder's optimal replication

Let $x_t := X^h_t + V^b_t(x_2)$ for all $t \in [0,T]$. We define $\mathcal{E}^h_{0,\tau}(x)$ as the unique $y \in \mathbb{R}$ such that there exists a strategy $\psi \in \Psi(y,-A)$ satisfying $V_\tau(y,\psi) = x_\tau$. One can check that

$$x_2 - \mathcal{E}^h_{0,\tau}(x) = \big\{ p \in \mathbb{R} \,:\, \exists\, \varphi \in \Psi(x_2 - p) \colon V_\tau(x_2 - p, \varphi) = x_\tau \big\}.$$

Assumption (Operator $\mathcal{E}^h$)

We assume that x is *replicable* for the holder, in the sense that for a given $x_2 \in \mathbb{R}$ and every $\tau \in \mathcal{T}$ there exist $(p,\psi) \in \mathbb{R} \times \Psi(x_2 - p, -A)$ such that $V_\tau(x_2 - p, \psi) = x_\tau$. Hence the operator $\mathcal{E}^h$ is well defined.

Definition (Holder's optimal replication)

We say that $(v^h_0(x_2), \tau^{*,h}) \in \mathbb{R} \times \mathcal{T}$ is a solution to the *holder's optimal replication problem* for $\mathrm{ACC}\,(A, X^h)$ if $v^h_0(x_2) = x_2 - \mathcal{E}^h_{0,\tau^{*,h}}(x)$ where $\tau^{*,h} \in \mathcal{T}$ is such that

$$\mathcal{E}^h_{0,\tau^{*,h}}(x) = \min_{\tau \in \mathcal{T}} \mathcal{E}^h_{0,\tau}(x).$$

Holder's acceptable price

Theorem (Holder's valuation)

Let the process x be right-upper-semicontinuous. If Assumptions (SFM) and (SBM) hold, then

- if $p \in \mathcal{H}^{a,h}(x_2)$, then $p < v_0^h(x_2)$ and thus
$$\widehat{p}^{f,h}(x_2) = v_0^h(x_2),$$
- $\mathcal{H}^{f,r,h}(x_2) \neq \emptyset$ and the holder's acceptable price satisfies
$$p^h(x_2) = v_0^h(x_2) = \widehat{p}^{f,r,h}(x_2) = \widehat{p}^{f,h}(x_2) = \widehat{p}^{s,h}(x_2),$$
- the stopping time $\tau^{*,h}$ is a holder's rational exercise time for (p^h, ψ^h).

- Let us stress that $p^i(x_1) \neq p^h(x_2)$, in general. Similarly, the issuer's break-even time $\tau^{*,i}$ and the holder's rational exercise time $\tau^{*,h}$ may fail to coincide in a nonlinear market.

BSDE Approach to American Options

BSDE Approach

Assume that the BSDE

$$\begin{cases} -dY_t = g(t, Y_t, Z_t)\,dt - Z_t\,d\mathcal{S}_t, \\ Y_T = X_T, \end{cases}$$

has a unique solution (Y, Z) in a suitable space of stochastic processes.

Definition (Conditional g-expectation $\mathcal{E}^g$)

If τ is a $\mathbb{G}$-stopping time and X_τ is a $\mathcal{G}_\tau$-measurable r.v., then the *$\mathcal{G}_t$-conditional g-expectation* $\mathcal{E}^g_{t,\tau}(X_\tau)$ equals Y where (Y, Z) is a unique solution to the BSDE

$$Y_t = X_\tau + \int_t^\tau g(u, Y_u, Z_u)\,du - \int_t^\tau Z_u\,d\mathcal{S}_u.$$

Definition (Monotonicity of $\mathcal{E}^g$)

The *monotonicity* of $\mathcal{E}^g$ holds if, for all $\tau \in \mathcal{T}$, if $X^1_\tau \geq X^2_\tau$ then $\mathcal{E}^g_{0,\tau}(X^1_\tau) \geq \mathcal{E}^g_{0,\tau}(X^2_\tau)$. The *strict monotonicity* of $\mathcal{E}^g$ holds if, for all $\tau \in \mathcal{T}$, if $X^1_\tau \geq X^2_\tau$ and $X^1_\tau \neq X^2_\tau$ then $\mathcal{E}^g_{0,\tau}(X^1_\tau) > \mathcal{E}^g_{0,\tau}(X^2_\tau)$.

RBSDE for a nonlinear optimal stopping problem

El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997) established the following result.

Proposition

Let (Y, Z, K^+) be the unique solution of the following RBSDE (g, X)

$$\begin{cases} -dY_t = g(t, Y_t, Z_t)\, dt + dK_t^+ - Z_t\, d\mathcal{S}_t, \\ Y_t \geq X_t, \quad Y_T = X_T, \quad \int_0^T (Y_t - X_t)\, dK_t^+ = 0, \end{cases}$$

then we have $Y_0 = \mathcal{V}_0^g(X)$ where

$$\mathcal{V}_0^g(X) = \sup_{\tau \in \mathcal{T}} \mathcal{E}_{0,\tau}^g(X_\tau).$$

Moreover, $\mathcal{V}_0^g(X) = \mathcal{E}_{0,\tau^*}^g(X_{\tau^*})$ where

$$\tau^* := \inf \{t \in [0, T] : Y_t = X_t\}.$$

Issuer's superhedging costs

- We fix $x_1 \in \mathbb{R}$ and we consider the unique solution (Y, Z, K^+) to the issuer's RBSDE (g, X) where the lower barrier is given by $X := V^b(x_1) - X^h$.

Proposition (Issuer's superhedging)

Let the monotonicity of the g-expectation hold. Then the lower bound for issuer's superhedging costs satisfies

$$\underline{p}^{s,i}(x_1) = Y_0 - x_1 = \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}\big(V^b_\tau(x_1) - X^h_\tau\big) - x_1$$

where (Y, Z, K^+) is a unique solution to the issuer's RBSDE (g, X) where $X := V^b(x_1) - X^h$.

Issuer's replication costs

Proposition (Issuer's replication)

Let the monotonicity of the g-expectation hold. Then:
(i) the pair (Y_0, Z) is an issuer's replicating strategy for $\mathrm{ACC}\,(A, X^h)$,
(ii) the issuer's minimum superhedging and replication costs satisfy

$$\widehat{p}^{\,r,i}(x_1) = \widehat{p}^{\,s,i}(x_1),$$

(iii) we have $\widehat{p}^{\,r,i}(x_1) = \mathcal{E}^g_{0,\tau^i}(X_{\tau^i})$ where

$$\tau^h := \inf\{t \in [0,T] : Y_t = X_t\}.$$

Proposition (Issuer's fair replication)

Let the strict monotonicity of the g-expectation hold. Then the issuer's fair replication price exists and satisfies

$$\widehat{p}^{\,r,i}(x_1) = \widehat{p}^{\,f,r,i}(x_1) = \widehat{p}^{\,f,i}(x_1).$$

Issuer's valuation through BSDE

Theorem (Issuer's BSDE)

(i) Let the monotonicity of the g-expectation hold. Then the lower bound for issuer's superhedging costs satisfies $\underline{p}^{s,i}(x_1) = Y_0 - x_1$ where (Y, Z, K^+) solves the RBSDE with the lower barrier $X = V^b(x_1) - X^h$

$$\begin{cases} -dY_t = g(t, Y_t, Z_t)\, dt + dK_t^+ - Z_t\, d\mathcal{S}_t, \\ Y_t \geq X_t, \quad Y_T = X_T, \quad \int_0^T (Y_t - X_t)\, dK_t^+ = 0, \end{cases}$$

The issuer's replicating strategy is given by the triplet $(Y_0 - x_1, \varphi^* = Z, \tau^{*,i})$ where

$$\tau^{*,i} := \inf\{t \in [0, T] : Y_t = X_t\}.$$

(ii) Let the strict monotonicity of the g-expectation hold. Then the issuer's maximum fair price for $\mathrm{ACC}\,(A, X^h)$ satisfies $\widehat{p}^{f,r,i}(x_1) = Y_0 - x_1 = \mathcal{V}_0^g(x) - x_1$ where

$$\mathcal{V}_0^g(x) = \sup_{\tau \in \mathcal{T}} \mathcal{E}_{0,\tau}^g(X_\tau).$$

Holder's valuation through BSDE

Theorem (Holder's BSDE)

(i) Let the monotonicity of the g-expectation hold. Then the upper bound for holder's superhedging costs satisfies $\overline{\mathrm{p}}^{s,h}(x_2) = x_2 - y_0$ where (y, z, k^-) solves the holder's RBSDE $(g, x = X^h + V^b(x_2))$

$$\begin{cases} -dy_t = g(t, y_t, z_t)\,dt - dk_t^- - z_t\,d\mathcal{S}_t, \\ y_t \le x_t, \quad y_T = x_T, \quad \int_0^T (x_t - y_t)\,dk_t^- = 0, \end{cases}$$

The holder's replicating strategy is given by the triplet $(x_2 - y_0, \psi^* = z, \tau^{*,h})$ where

$$\tau^{*,h} := \inf\{t \in [0,T] : y_t = x_t\}.$$

(ii) Let the strict monotonicity of the g-expectation hold. Then the holder's minimum fair price for $\mathrm{ACC}\,(A, X^h)$ satisfies $\widehat{\mathrm{p}}^{f,r,h}(x_2) = x_2 - y_0 = x_2 - v_0^g(x)$ where

$$v_0^g(x) = \inf_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}(x_\tau).$$

Holder's rational exercise time

Definition (Rational exercise time)

A $\mathbb{G}$-stopping time $\tau^h \in \mathcal{T}$ is a *rational exercise time* if an American contract is traded at the price $p = \underline{p}^{r,h}$ at time 0 and there exists a strategy $\psi \in \Psi(x_2 - p)$ such that $V_{\tau^h}(x_2 - p, \psi) \geq x_{\tau^h}$. If the strict monotonicity of the g-expectation holds, then the inequality $V_{\tau^h}(x_2 - p, \psi) \geq x_{\tau^h}$ can be replaced by the equality.

Proposition

If the strict monotonicity of the g-expectation holds, then a holder's rational exercise time $\widehat{\tau}$ is characterized by the conditions:
(i) $y^{\widehat{\tau}}$ is a g-martingale, that is, $k_{\widehat{\tau}} = 0$,
(ii) the equality $y_{\widehat{\tau}} = x_{\widehat{\tau}}$ holds.
The earliest rational exercise time equals $\tau^h := \inf\{t \in [0,T] : y_t = x_t\}$.

- In any linear market model (but not in a general nonlinear model), any holder's rational exercise time τ^c is also an issuer's break-even time τ^h.