ROM Simulation with Exact Means, Covariances, and Multivariate Skewness

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Motivation

- In many areas of finance (e.g., option pricing, risk management), numerical methods are required
- Example: simulation of (discrete) return distributions
- Generally desirable: “good fit” between simulated returns and some pre-specified distribution (e.g., Binomial model, number of time steps)

Motivation

- One approach to achieve “good fit”: impose restrictions that ensure matching the first few moments of some pre-specified distribution
- For univariate distributions: easy. For multivariate distributions: Hoyland/Kaut/Wallace (2003), but confined to covariance matrix and higher marginal moments.
- What about higher moments beyond marginal moments? In particular, what about multivariate skewness?
- Downside risk. . . increased correlations in times of crises. . . many asset classes affected!

Motivation

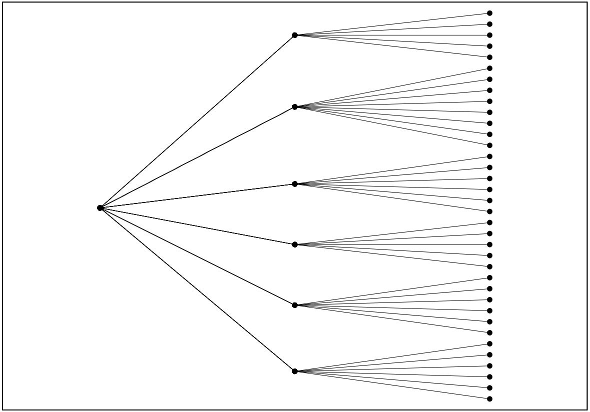
- Goal: generate discrete samples of multivariate distributions (risk factors, asset returns,...)
- For n assets/risk factors with expected (excess) returns $\boldsymbol{\mu}_n$, covariance matrix $\mathbf{S}_n$, and m different states of nature, find

$$\mathbf{X}_{mn} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix}$$

such that

$$m^{-1}(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n')'(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n') = \mathbf{S}_n.$$

Motivation



Motivation

- Basis for our approach: Ledermann et al. (2011) (ROM simulation – multivariate samples matching pre-specified means and covariances)
- Additional requirements:
  - If $\mathbf{X}_{mn}$ represents asset returns, make sure that they do not allow for arbitrage opportunities $\rightarrow$ "No-arbitrage ROM simulation", Geyer/Hanke/Weissensteiner (JEDC, 2014)
  - If $\mathbf{X}_{mn}$ should not only have pre-specified $\boldsymbol{\mu}_n$ and $\mathbf{S}_n$, but also "the correct skewness" $\rightarrow$ this paper. Correct skewness may be important for both (non-traded) risk factors and returns of (traded) assets

ROM Simulation (Ledermann et al., 2011)

- n assets with expected (excess) returns $\boldsymbol{\mu}_n$ and covariance matrix $\mathbf{S}_n$
- Goal: generate a sample $\mathbf{X}_{mn}$ of m observations on the n random variables such that

$$m^{-1}(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n')'(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n') = \mathbf{S}_n. \qquad (1)$$

- $\mathbf{S}_n$ can be decomposed (since pos. semi-def.) into $\mathbf{S}_n = \mathbf{A}_n'\mathbf{A}_n$ (using, e.g., Cholesky decomposition)

ROM Simulation

- Defining

$$\mathbf{L}_{mn} = m^{-1/2}(\mathbf{X}_{mn} - \mathbf{1}_m \boldsymbol{\mu}_n')\mathbf{A}_n^{-1}, \tag{2}$$

solving (4) is equivalent to finding a matrix $\mathbf{L}_{mn}$ satisfying

$$\mathbf{L}_{mn}'\mathbf{L}_{mn} = \mathbf{I}_n \quad \text{with} \quad \mathbf{1}_m'\mathbf{L}_{mn} = \mathbf{0}'. \tag{3}$$

- Ledermann (2011) call solutions to eq. (3) *L matrices*

Mechanics of ROM simulation

- In general: pre-multiply an L matrix by a permutation matrix and post-multiply this product by any square orthogonal matrix $\mathbf{R}_n$
- Pre-multiplication is primarily for controlling the time-ordering of random samples (not relevant here)
- The basis for Geyer et al. (2014) and for this paper is the following simplified version:

$$\mathbf{X}_{mn} = \mathbf{1}_m \boldsymbol{\mu}_n' + \sqrt{m}\mathbf{L}_{mn}\mathbf{R}_n\mathbf{A}_n \tag{4}$$

ROM simulation

- Since we will frequently need the scaled L matrix with column variance equal to 1, we define $\mathbf{L} = \sqrt{m}\mathbf{L}_{mn}$
- Ledermann et al. (2011) suggest using matrices $\mathbf{R}_n$ representing randomized rotation angles and directions
- Main insight of Geyer et al. (2014): “wise choice” of rotation directions combined with restricted intervals for random rotation angles ensures absence of arbitrage

Multivariate higher moments

- So far, multivariate skewness and kurtosis measures are not very common in finance
- Recently, skewed multivariate distributions received increased attention in financial modeling
- Most frequently used in the literature: Mardia (1970) skewness and kurtosis measures
- Criticized by Kollo (2008) for being overly aggregated/simplistic
- Kollo (2008) develops informationally richer measures for multivariate skewness and kurtosis

Co-skewness matrix

- Given n asset returns $\mathbf{r} = (r_1, \ldots, r_n)'$, with means $\bar{\mathbf{r}}$ and covariance matrix Σ, their $(n \times n^2)$ co-skewness matrix $\mathbf{M}_3$ can be defined as follows:

$$\mathbf{M}_3 = [\mathbf{D}_1 | \mathbf{D}_2 | \ldots | \mathbf{D}_n] \,, \tag{5}$$

where

$$\mathbf{D}_i = \begin{bmatrix} d_{i11} & d_{i12} & \ldots & d_{i1n} \\ d_{i21} & d_{i22} & \ldots & d_{i2n} \\ \vdots & & \ddots & \vdots \\ d_{in1} & d_{in2} & \ldots & d_{inn} \end{bmatrix}, \tag{6}$$

$$d_{ijk} = \mathbb{E}[\mathbf{r}_i^* \mathbf{r}_j^* \mathbf{r}_k^*], \tag{7}$$

$$\mathbf{r}^* = \Sigma^{-1/2}(\mathbf{r} - \bar{\mathbf{r}}). \tag{8}$$

Multivariate skewness measures

- Using the entire co-skewness matrix $\mathbf{M}_3$ is impractical (n^3 elements).
- Multivariate skewness measures aggregate the information contained in $\mathbf{M}_3$.
- In this aggregation, some information contained in $\mathbf{M}_3$ gets lost.
- There is no universal best way to construct a multivariate skewness measure.
- In finance and financial risk management, retaining directional information is particularly desirable.

Mardia's skewness measure

- In terms of the co-skewness matrix $\mathbf{M}_3$, Mardia's skewness measure is a scalar:

$$\tau_M(\mathbf{M}_3) = \sum_{ijk} d_{ijk}^2, \tag{9}$$

  with d_{ijk} as defined in equation (7).
- The resulting skewness value is a scalar, which may be identical for distributions of very different shape.
- Mardia's skewness (and kurtosis) measures are criticized by Kollo (2008) based on an analysis of their shortcomings in Gutjahr (1999).
- Adding to this list, Mardia's skewness measure disregards the sign of co-skewness terms (!)

Kollo's skewness measure

- Kollo (2008) defined an alternative, richer measure of skewness:

$$\mathbf{b}(\mathbf{M}_3) = \begin{pmatrix} \sum_{i1k} d_{i1k} \\ \sum_{i2k} d_{i2k} \\ \vdots \\ \sum_{ink} d_{ink} \end{pmatrix}, \quad (10)$$

  with d_{ijk} as defined in equation (7).
- The resulting skewness value is a vector, not a scalar as in the case of Mardia's skewness.
- In most cases, Kollo's skewness measure retains more information compared to Mardia's skewness when aggregating co-skewness terms.

Mardia and Kollo skewness of samples generated using ROM simulation

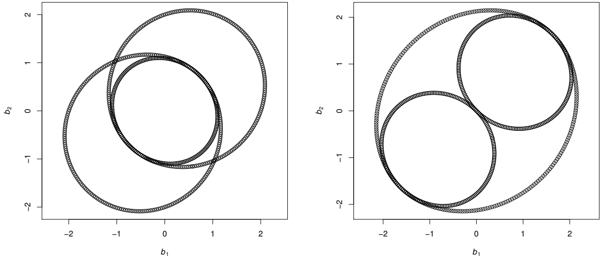


Figure : Examples of attainable Kollo skewness vectors $\mathbf{b} = (b_1, b_2)'$ for m=4 and n=2 using two different L matrices. The Mardia skewness of the first matrix is 2/3, and that of the second matrix is 3.

What values of Kollo skewness are attainable?

- The maximum norm of the Kollo skewness (when using the distance-of-one-vertex-maximizing simplex described in Geyer et al., 2014) is attained when each element of the skewness vector **b** is given by

$$b^* = (m-2)\sqrt{n/(m-1)}, \tag{11}$$

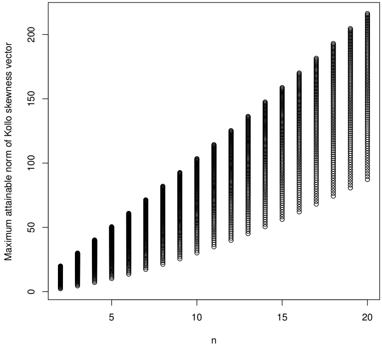
which results in a norm of

$$||\mathbf{b}^*|| = \frac{n(m-2)}{\sqrt{m-1}}. \tag{12}$$

- This relation provides an additional lower bound for the number of states to be used for ROM simulation.

What values of Kollo skewness are attainable?

- Fig. 2 shows max. attainable norms of Kollo skewness vectors for different dimensions (m, n). $m = n+2, \ldots, n+100$.



ROM simulation matching also multivariate skewness

- Let us assume that a given skewness vector $\mathbf{b} = (b_1, b_2, \ldots, b_n)'$ is attainable
- Let $\mathbf{L} = \sqrt{m}\mathbf{L}_{m,n}$ be a (scaled) L matrix with Kollo skewness $\mathbf{b}$
- Use $m \geq 4$ (and also $m \geq n + 2$) as a minimal condition for the sample size
- Recall that

$$\mathbf{1}'_m\mathbf{L} = 0 \text{ and } \mathbf{L}'\mathbf{L} = m\mathbf{I}_n \tag{13}$$

ROM simulation matching also multivariate skewness

- The problem of finding $\mathbf{L}^*$ with a pre-specified Kollo skewness vector $\mathbf{b}^*$ can be expressed as a system of linear, quadratic, and cubic equations
- This system can be simplified to finding the roots of one cubic equation, followed by solving a pair of linear and quadratic equations
- For details, see Section 4 of the paper

Computation times

	n			
m	5	10	50	100
$n+2$	0.01	0.03	0.75	5.19
	(0.00)	(0.01)	(0.03)	(0.21)
$2n$	0.01	0.05	2.54	17.91
	(0.00)	(0.01)	(0.03)	(0.35)
$3n$	0.01	0.05	2.66	20.55
	(0.00)	(0.01)	(0.09)	(0.84)
$4n$	0.01	0.05	2.89	20.61
	(0.00)	(0.01)	(0.07)	(0.38)

Table : Average computation time in seconds (standard deviation in brackets) required to simulate m observations on n random variables with a random target Kollo skewness vector. Averages and standard deviations have been computed from 10 random vectors per problem size (m, n).

Conclusion

- Extension of original ROM simulation: Match also Kollo skewness in addition to means and covariances
- Potential applications: Large-scale risk management simulations (banks), other problems in finance
- No-arbitrage can be addressed by combining theoretical results on required sample size (Geyer et al., 2014) with “check – discard – resample”-loop
- Algorithm is very fast – for a given number of random variables (e.g., risk factors), computation time increases only slowly in the number of observations
- Further research: extension of the algorithm to match also Kollo kurtosis.

Publication details

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No-arbitrage ROM simulation

- L matrices as defined before have zero mean
- $\mathbf{Y}_{mn} = \mathbf{X}_{mn} - \mathbf{1}_m \boldsymbol{\mu}'_n$ will be important, which can be computed from $\mathbf{L}_{mn}$ using eq. (2):

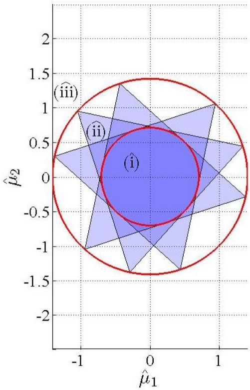
$$\mathbf{Y}_{mn} = \sqrt{m}\mathbf{L}_{mn}\mathbf{A}_n \equiv \mathbf{L}\mathbf{A}_n \tag{14}$$

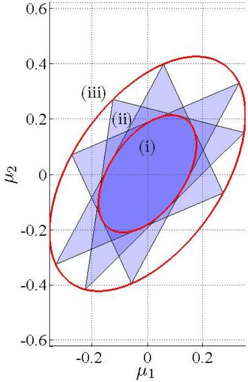
- $\mathbf{Y}_{mn}$ is linked to $\mathbf{L}_{mn}$ by a particular affine transformation $\mathcal{A}(\cdot)$, $\mathbf{Y}_{mn} = \mathcal{A}(\mathbf{L}_{mn})$
- $\mathbf{Y}_{mn}$ can be interpreted as a sample of asset returns with the correct covariance structure $\mathbf{S}_{mn}$ and means of $\mathbf{0_n}$

No-Arbitrage ROM Simulation (Geyer et al., 2014)

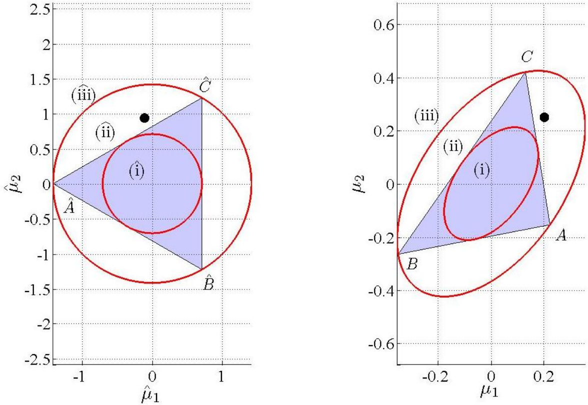
- Geometric interpretation of L matrices: Rows of $-\mathbf{L}_{mn}$ define a simplex (can be constructed deterministically)
- This simplex is regular if $m=n+1$ ("complete market" with n risky assets and one risk-free asset), and irregular if $m>n+1$ ("incomplete market")
- Multiplying the simplex by $\mathbf{R}_n$ rotates the simplex
- Absence of arbitrage means that expected excess returns μ_n are inside the simplex
- Key insight: $\mathbf{R}_n$ can be chosen judiciously to ensure that μ_n is inside the simplex

Two-dimensional case

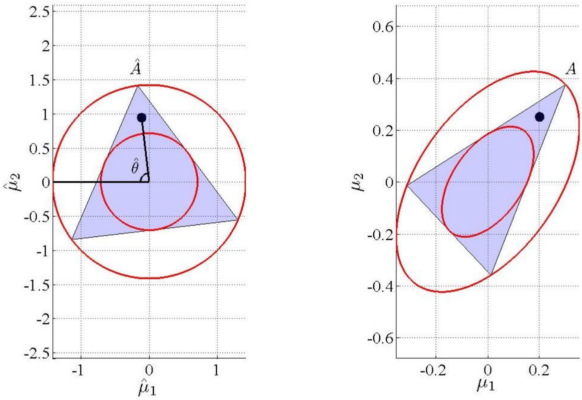




Two-dimensional case



Two-dimensional case



Generalization to *n* dimensions

- Equilateral triangle changes to a regular *n*-simplex
- In- and circumcircles of the triangle become hyperspheres, whose images are hyperellipsoids
- Deterministic construction of the simplex easily generalizes to *n* dimensions
- Rotation in *n* dimensions is a bit more tricky...