A more meaningful parameterisation of the Lee-Carter model

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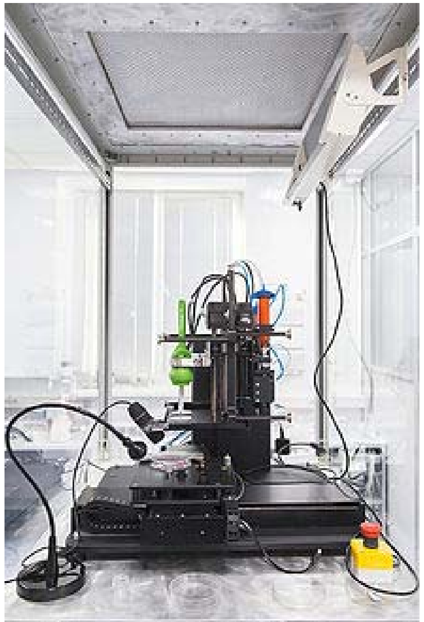
Life expectancy is evolving

BBC NEWS

US life expectancy declines for first time in 20 years
8 December 2016

HUMAN LONGEVITY, INC.



"Fixing the 'problem' of ageing is the mission of Silicon Valley, where billions is pouring into biotech firms working to 'hack the code' of life"

the **guardian** Jan 11 2015

The Obsession With 'Curing' Aging Is Now Big Business
FORTUNE Magazine, MARCH 7 2016

Rise in life expectancy has stalled since 2010, research shows

18 July 2017 theguardian

BREAKTHROUGH PRIZE

“Fiery debate” between experts

“the steady rise in life expectancy during the past two centuries may soon come to an end” (Olshansky et al. 2005)

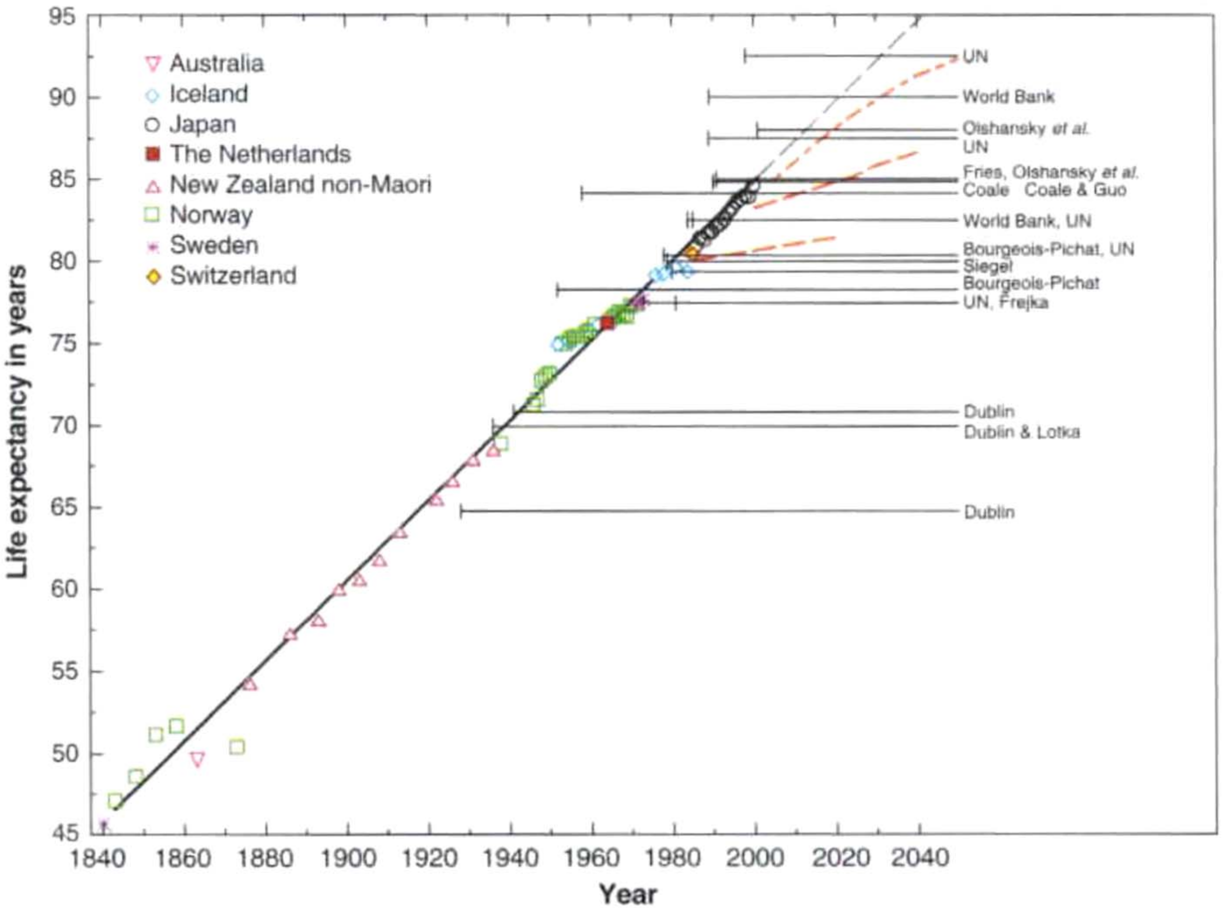
Vaupel et al. (2009): *“most babies born since 2000 in countries with long-lived residents will celebrate their 100th birthdays”*

UK Longevity Science Panel (2014). *What is ageing? Can we delay it?*

Olshansky, S. Jay, et al. (2005).A potential decline in life expectancy in the United States in the 21st century." *New England Journal of Medicine* 352.11 1138-1145.

Christensen, Doblhammer, Rau, Vaupel (2009). Ageing populations: the challenges ahead. *The Lancet*, *374*(9696).

Life expectancy improvements have been significant and have been underestimated



"experts have repeatedly asserted that life expectancy is approaching a ceiling: these experts have repeatedly been proven wrong."

"[*forecast limits between*] *1928* [*and*] *1990 have been broken, <u>on average 5 years after publication</u>*"

"the ignominious saga of life-expectancy maxima"

Source: Oeppen, J. and Vaupel, J. (2002). Broken Limits to Life Expectancy, *Science*, 296, 1029-31.

Widespread use of Lee-Carter-type methods

- Much recent interest in methods of mortality forecasting
- Lee-Carter-type methods offer a number of advantages in forecasting, and have become dominant (Deaton and Paxson, 2004).
- But could they be reformulated in such a way as to be more useful for mortality modelling and comparison, in addition to forecasting?

→ We will present two refinements to the Lee-Carter model to make it more interpretable, and able to be used for mortality analysis and comparison

Deaton, A. S. and Paxson, C. (2004). Mortality, income, and income inequality over time in Britain and the United States. In *Perspectives on the Economics of Aging*, pages 247{286. University of Chicago Press.

Outline

- Key features of the Lee-Carter model
- Reformulated and normalised model:
  - Choice of constraints
  - “Needed exposure”
  - Interpretation of parameters of original versus reformulated model
- Application to international data
- Conclusions

Key features of the Lee-Carter model

Lee-Carter model – key features

$$\ln m_{xt} = a_x + b_x k_t + \varepsilon_{xt} \qquad x = 0,1,...,p, \quad t = 1,2,...,N$$

- m_{xt} is observed central mortality rate at age x in year t
- a_x is an age-related parameter (age "intercept" term)
- k_t is an index of overall mortality level in year t ("trend" component)
- b_x reflects responsiveness of age x to changes in the overall mortality level (age "response to trend" component)
- ε_{xt} is error – assume uncorrelated, zero mean, constant variance

Model summarises mortality patterns and trends. Forecasts are produced by using time series methods to forecast k_t.

Lee, R. and Carter, N. (1992). Modeling and forecasting US mortality. *Journal of the American Statistical Association*, 87(419), 659-671.

Lee-Carter model – identification problem

- Lee-Carter model has a well-known "identification problem".
- A change in "scale" of k_t can be "taken up" in b_x:

$$a_x + \sigma b_x \times \frac{k_t}{\sigma} = a_x + b_x k_t$$

- A "shift" in k_t can be "taken up" in a_x:

$$\left(a_x + \mu b_x\right) + b_x\left(k_t - \mu\right) = a_x + b_x k_t$$

- In general,

$$\left(a_x + \frac{\mu}{\sigma}\sigma b_x\right) + \sigma b_x \frac{\left(k_t - \mu\right)}{\sigma} = a_x + b_x k_t$$

- Lee-Carter solves the identification problem by applying constraints:

$$\sum\nolimits_t k_t = 0 \text{ and } \sum\nolimits_x b_x = 1$$

Is this the best option?

Lee-Carter model – usual normalisation

With $\sum_t k_t = 0$ and $\sum_x b_x = 1$

$$\sum_t \ln m_{xt} = Na_x + b_x \sum_t k_t = Na_x \,,\ a_x = \frac{\sum_t \ln m_{xt}}{N}$$

a_x is interpretable as average (across-time) log-mortality at age x

$$\sum_x \left(\ln m_{xt} - a_x\right) = k_t \sum_x b_x = k_t$$

k_t is (somewhat) interpretable as the sum across age of

log-mortality less average (across-time) log-mortality at age x

b_x reflects response at x to changes in k_t - not directly interpretable

A different normalisation would not affect model and forecasts but

may give more interpretable and comparable parameter values

Lee-Carter model – modelled quantity

- People have difficulty interpreting probabilities, and interpret very low probabilities as essentially zero (Stone et al. 1994, Kunreuether et al. 2001).
- "*Even an expert in the mortality field has difficulty interpreting the meaning of an improvement in the mortality rate*" (Pollard, 1988)
- Define n_{xt} as the "*needed exposure*" (NE) at age x in year t to generate one expected death

$$n_{xt} = 1 / m_{xt}$$

e.g. if $m_{xt} = 0.002$ then $n_{xt} = 500$

- Related to number "*needed-to-treat*" in medicine (Laupacis et al. 1988)
- We cast the Lee-Carter model in terms of NE

Stone, E. R., Yates, J., and Parker, A. M. (1994). Risk communication: Absolute versus relative expressions of low-probability risks. *Organizational Behavior and Human Decision Processes*, 60(3):387-408. Kunreuther, H., Novemsky, N., and Kahneman, D. (2001). Making low probabilities useful. *Journal of Risk and Uncertainty*, 23(2):103-120. Pollard, J. H. (1988). On the decomposition of changes in expectation of life and differentials in life expectancy. *Demography*, 25(2):265-276. Laupacis, A., Sackett, D. L., and Roberts, R. S. (1988). An assessment of clinically useful measures of the consequences of treatment. *New England Journal of Medicine*, 318(26):1728-1733.

Reformulated and normalised model

Reformulated and normalised Lee-Carter

de Jong, Tickle and Xu (2016)

1. Replace m_{xt} by $1/n_{xt}$:

$$\ln n_{xt} = -a_x - b_x k_t - \varepsilon_{xt}$$

2. Write

$$k_t = \mu - \sigma \ln n_t \Leftrightarrow \ln n_t = -\frac{k_t - \mu}{\sigma} \Leftrightarrow n_t = e^{-(k_t - \mu)/\sigma}$$

3. Choose scale σ and shift μ so that $\ln n_t$ is meaningful and practical

$$\text{e.g. } \min_{\mu,\sigma} \sum_{x,t} w_{xt} \left(\ln n_{xt} - \ln n_t \right)^2$$

where weights $w_{xt} \geq 0$ are used to emphasise certain ages, times.

de Jong, P., Tickle, L., Xu, J. (2016). A Transparent parametrization of the Lee–Carter model based on "Needed Exposure". *Macquarie University Centre for Financial Risk Working Paper 16-01.*

Reformulated and normalised Lee-Carter

de Jong, Tickle and Xu (2016)

4. Solve minimisation

   Since k_t is independent of age x the minimisation problem reduces to

   $$\text{e.g. } \min_{\mu,\sigma} \sum_t w_t \left\{ \varepsilon\left(\ln n_{xt}\right) - \frac{\mu - k_t}{\sigma} \right\}^2$$

   k_t is from SVD or Poisson regression or ...
   $\varepsilon\left(\ln n_{xt}\right)$ is the weighted average of $\ln n_{xt}$ at t, $w_t = \sum_x w_{xt}$.

   Weights may be chosen to be from a standard population in a certain year, to enable comparison of parameters across populations and over time.

de Jong, P., Tickle, L., Xu, J. (2016). A Transparent parametrization of the Lee–Carter model based on "Needed Exposure". *Macquarie University Centre for Financial Risk Working Paper 16-01.*

Reformulated and normalised Lee-Carter

de Jong, Tickle and Xu (2016)

5. Then

$$\ln n_{xt} = \alpha_x + \beta_x \ln n_t + \varepsilon_{xt}$$

where

$$\alpha_x = -a_x - \mu b_x$$

$$\beta_x = \sigma b_x$$

$$\ln n_t = -\frac{k_t - \mu}{\sigma}.$$

Parameters μ and σ have been chosen so that n_t is interpretatable as an appropriate average / overall measure of n_{xt} at time t.

de Jong, P., Tickle, L., Xu, J. (2016). A Transparent parametrization of the Lee–Carter model based on "Needed Exposure". *Macquarie University Centre for Financial Risk Working Paper 16-01.*

Interpretation of fitted n_t

$$\hat{n}_t = e^{-\frac{(k_t - \hat{\mu})}{\hat{\sigma}}}$$

μ and σ chosen so that $\hat{n}_t$ represents overall level of mortality at time t

$$\hat{n}_t = \prod_x \frac{1}{\hat{m}_{xt}^{w_{xt}}}$$

i.e. NE to get one expected death where the one-year mortality
is the weighted geometric mean (across age) of the m_{xt}, $\sum_x w_{xt} = 1$.

Actuarial interpretation of $\hat{n}_t$: the total amount payable per death over the year t (ignoring interest) if everyone in the population contributes at the rate of $1 p.a. while alive over the year t to a "term insurance" fund

de Jong, P., Tickle, L., Xu, J. (2016). A Transparent parametrization of the Lee–Carter model based on "Needed Exposure". *Macquarie University Centre for Financial Risk Working Paper 16-01.*

Interpretation of fitted n_{xt} and β_x

$$\ln \hat{n}_{xt} = \alpha_x + \beta_x \ln \hat{n}_t$$

So

$$\beta_x = \frac{\partial \ln \hat{n}_{xt}}{\partial \ln \hat{n}_t}$$

is the percentage change in NE at age *x* given a one percent change in $\hat{n}_t$
i.e. the "elasticity" at age *x*

$\beta_x > 1$ indicates greater responsiveness to mortality change

$\beta_x < 1$ indicates lesser responsiveness to mortality change

$\ln \hat{n}_{xt} = \alpha_x + \beta_x \ln \hat{n}_t$ is an age-specific transform of common $\hat{n}_t$

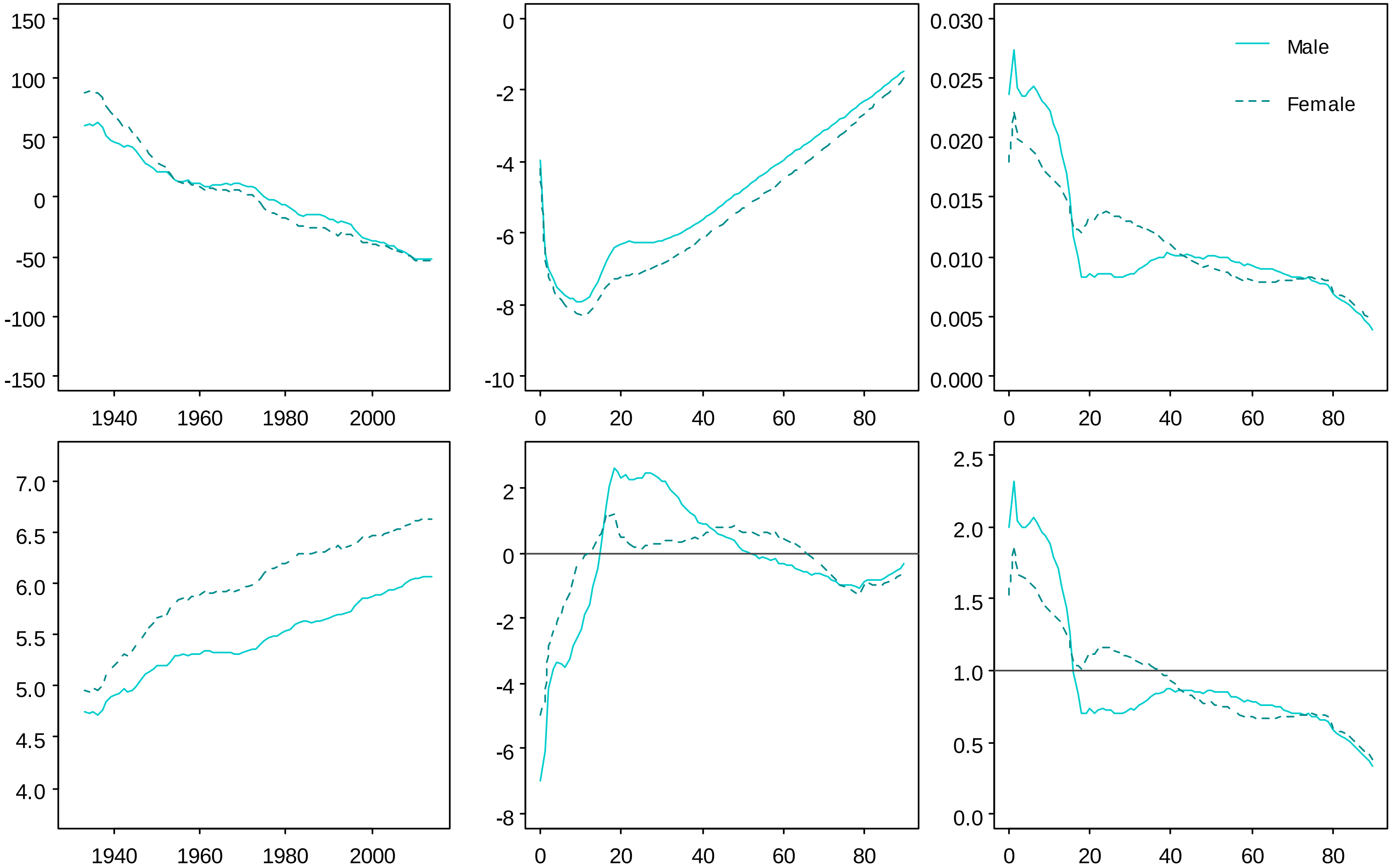
α_x adjusts the level of NE at age *x*, analagous to a regression intercept

de Jong, P., Tickle, L., Xu, J. (2016). A Transparent parametrization of the Lee–Carter model based on "Needed Exposure". *Macquarie University Centre for Financial Risk Working Paper 16-01.*

Application to international data

Application to United States data

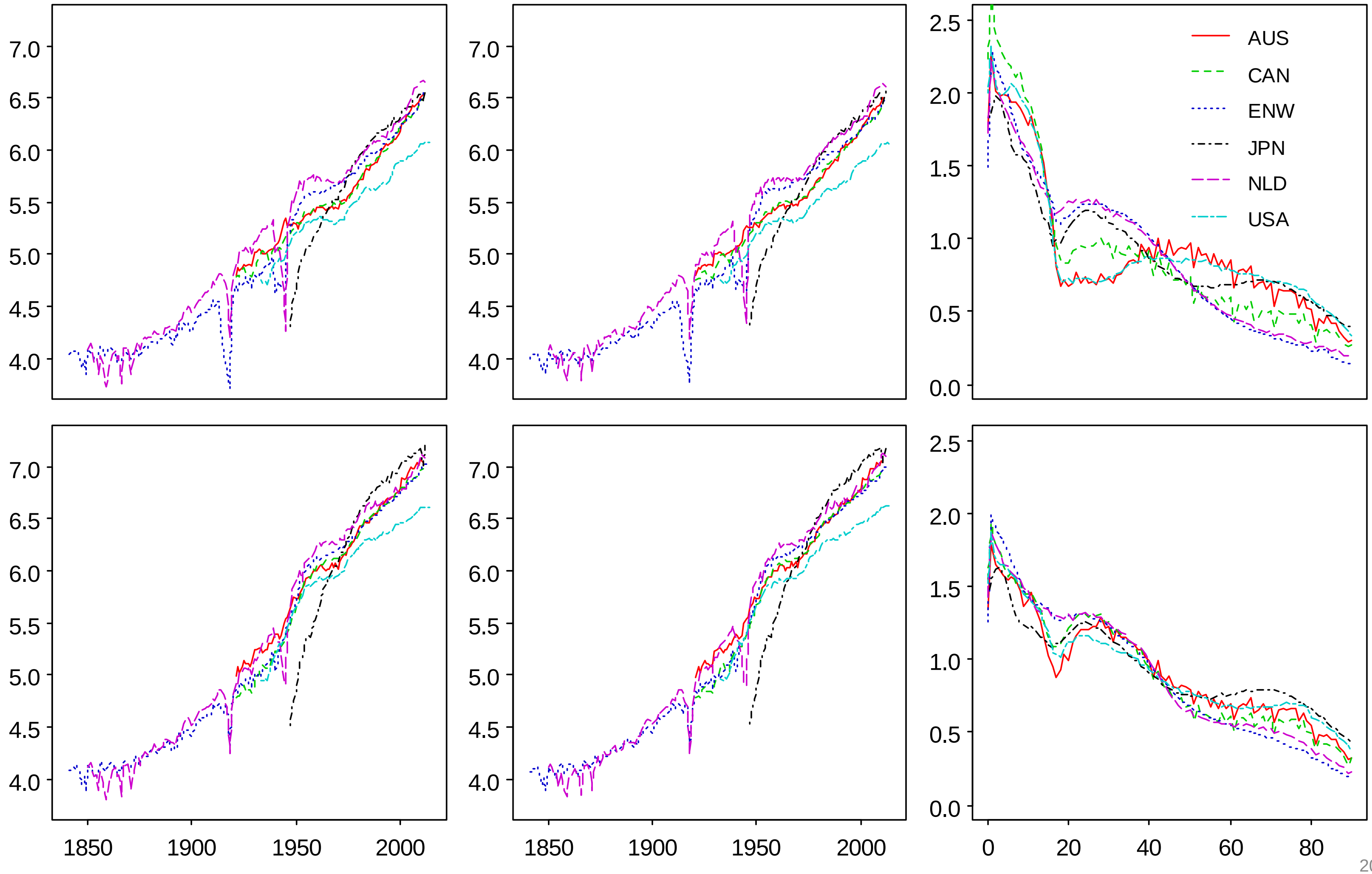
LC (top) and reformulated LC (bottom), k_t/lnn_t (left), a_x/α_x (middle), b_x/β_x (right)



Source: Authors own calculations. Data from Human Mortality Database <www.mortality.org>

International comparison

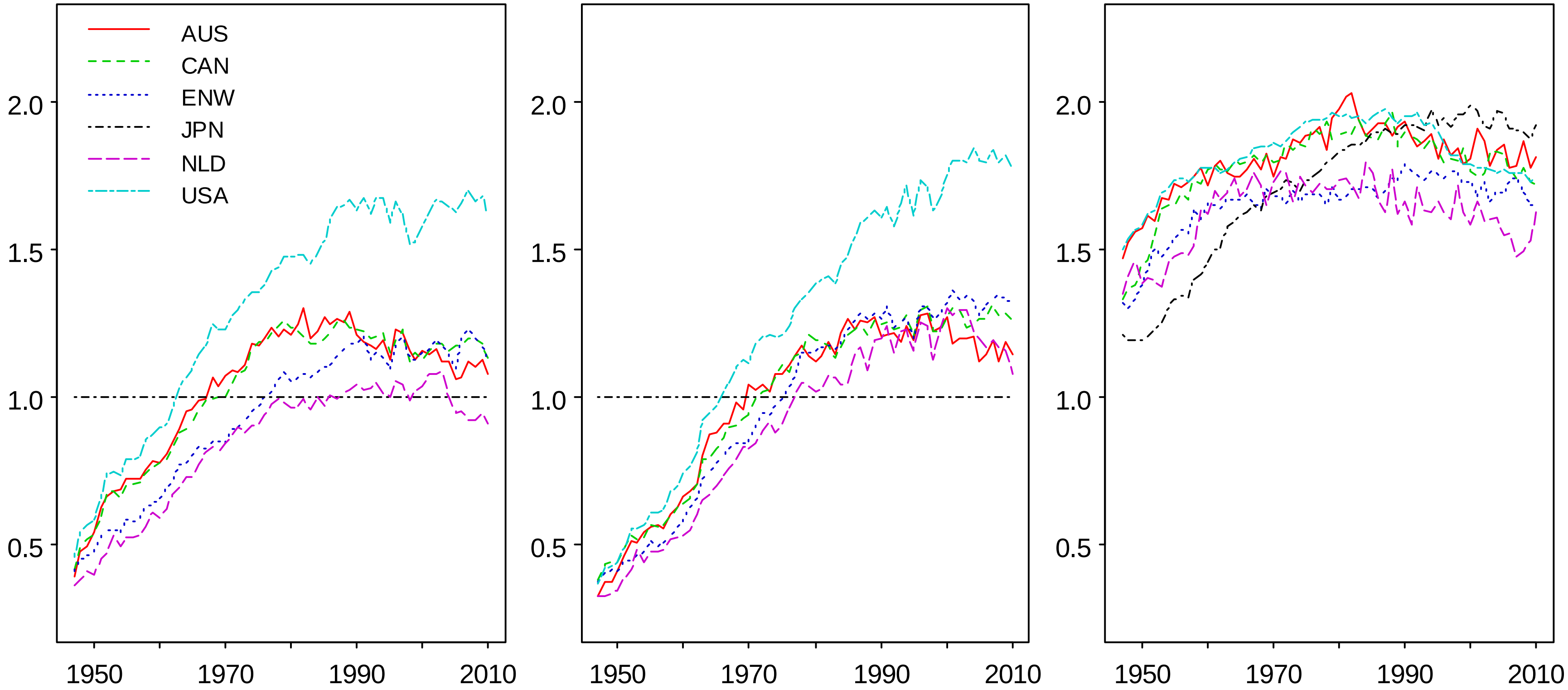
Observed lnGM (left), lnn_t (middle), b_x (right), for males (top) females (bottom)



Source: de Jong, Tickle and Xu (2016)

Cross-country and male/female comparisons

Males relative to Japan (left), females relative to Japan (middle), sex ratio (right)



Source: de Jong, Tickle and Xu (2016)

Conclusions

Conclusions

- Standard LC normalisation:
  - Block to interpretation and across-population comparison
- New interpretable formulation and normalisation:
  - Transformed time series component $\ln n_t$ is directly interpretable in terms of mortality improvement (n_t approximates weighted geometric mean of n_{xt})
  - Transformed age response component β_x is directly interpretable as elasticity
  - With common weights, results are comparable over time and across populations
  - Ratios of n_t across populations are valid measures of relative mortality
  - Forecasts remain identical to original LC
  - Trivial to implement

Questions?