Market Efficiency and the Growth Optimal Portfolio

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Outline

1. Growth Optimal Portfolio (GP)
2. Efficient Market Property (EMP)
3. Proxy of the GP: Hierarchically Weighted Index (HWI)
4. Empirical study of EMP
5. Hierarchical Stock Market Model (HMM): HWI equals GP
6. Diversification Theorem

Market Setting

- m stocks $S_t = (S_t^1, \dots, S_t^m)^\top$

$$\frac{dS_t}{S_t} = a_t dt + b_t \cdot dW_t$$

- portfolio value V_t^π

$$\frac{dV_t^\pi}{V_t^\pi} = \pi_t^\top \cdot \frac{dS_t}{S_t} = \pi_t^\top \cdot a_t dt + \pi_t^\top \cdot b_t \cdot dW_t$$

$$\pi_t^\top \cdot \mathbf{1} = 1$$

Growth Optimal Portfolio (GP)

- GP: **log-utility maximizing portfolio**
  Kelly (1956), MacLean et al. (2011)
- **maximizes** in the long run **long-term growth rate** (GR):

$$G_T^\pi = \frac{1}{T} \ln \left(\frac{V_T^\pi}{V_0^\pi} \right)$$

- **performance measure** distinguishes between proxies of GP

GP Theorem, Filipović & Platen (2009)

SDE of GP:

$$\frac{dV_t^{\pi_*}}{V_t^{\pi_*}} = \lambda_t dt + \theta_t^\top \cdot (\theta_t dt + dW_t) \tag{1}$$

$$\theta_t = b_t^\top \cdot \pi_t^*, \tag{2}$$

π_t^* and λ_t solution of

$$\begin{pmatrix} b_t b_t^\top & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{pmatrix} \begin{pmatrix} \pi_t^* \\ \lambda_t \end{pmatrix} = \begin{pmatrix} a_t \\ 1 \end{pmatrix} \tag{3}$$

Sufficient condition: invertibility of $b_t \cdot b_t^\top$

Efficient Market Property (EMP)

- **benchmarked portfolio value:** $\hat{V}_t^\pi = \frac{V_t^\pi}{V_t^{\pi^*}}$
- **return process:**

$$d\hat{Q}_t^\pi = \frac{d\hat{V}_t^\pi}{\hat{V}_t^\pi} = \left(\pi_t^\top \cdot b_t - \theta_t^\top\right) \cdot dW_t$$

Theorem

Instantaneous expected returns of benchmarked portfolios equal zero.

EMP Test

- Fatou's lemma $\rightarrow$ nonnegative local martingale is supermartingale $\rightarrow$

$$E_t(\hat{V}^\pi_{t+h}) \leq \hat{V}^\pi_t$$

- $$E_t\left(\frac{\hat{V}^\pi_{t+h} - \hat{V}^\pi_t}{\hat{V}^\pi_t}\right) \leq 0$$

- deep link between GP and market efficiency

Efficient Market Property

Nonnegative benchmarked portfolios have zero or negative expected returns over any time period.

EMP in the spirit of Fama (1970, 1991, 1998)

Constructing proxy of GP

- **equally-weighted approach** seems to do well, DeMiguel et al. (2009)
- we go beyond naive diversification, **capture** economic **dependencies**
- **classification of economic activities**
- naive diversification within each group of the hierarchy
- **Hierarchically Weighted Index** (HWI)

Data: Datastream stock lists

Country	Active	No. Active	Downl.	Dead	No. Dead	Downl.	Removed.
CANADA	LTOTMKCN	250	245	DEADCN1-2	6814	5310	598
UNITED STATES	LTOTMKUS	**999**	998	DEADUS1-6	22189	17009	467
HONG KONG	LTOTMKHK	130	127	DEADHK	248	200	8
JAPAN	LTOTMKJP	**1000**	1000	DEADJP	1681	1569	14
UNITED KINGDOM	LTOTMKUK	**549**	539	DEADUK	5625	5263	1037
SPAIN	LTOTMKES	120	116	DEADES	264	180	9
NETHERLANDS	LTOTMKNL	117	107	DEADNL	429	343	39
AUSTRALIA	LTOTMKAU	160	160	DEADAU	1784	1531	32
SWITZERLAND	LTOTMKSW	150	146	DEADSW	360	246	12
BELGIUM	LTOTMKBG	90	90	DEADBG	271	245	23
FRANCE	LTOTMKFR	250	247	DEADFR	1534	1400	243
GERMANY	LTOTMKBD	250	235	DEADBD	3000	2229	21
ITALY	LTOTMKIT	160	150	DEADIT	422	339	24
SINGAPORE	LTOTMKSG	100	100	DEADSG	409	391	4
NORWAY	LTOTMKNW	50	50	DEADNW	415	400	34
IRELAND	LTOTMKIR	37	37	DEADIR	129	108	22
SWEDEN	LTOTMKSD	70	62	DEADSD	819	709	72
FINLAND	LTOTMKFN	50	47	DEADFN	149	124	17
AUSTRIA	LTOTMKOE	50	49	DEADOE	196	160	8
PORTUGAL	LTOTMKPT	50	48	DEADPT	239	165	51
DENMARK	LTOTMKDK	50	47	DEADDK	277	254	15
NEW ZEALAND	LTOTMKNZ	50	50	DEADNZ	252	224	4
ISRAEL	LTOTMKIS	50	49	DEADIS	505	413	1

Base dates, no. of stocks and type of industrial grouping

Country	HWI Country Base Date	No. of Stocks	Industrial Grouping
CANADA	01/01/1990	245	sector
UNITED STATES	02/01/1984	998	subsector
HONG KONG	01/01/1986	127	sector
JAPAN	01/01/1990	1000	subsector
UNITED KINGDOM	01/01/1985	539	sector
SPAIN	03/01/2000	116	sector
NETHERLANDS	01/01/1990	107	sector
AUSTRALIA	01/01/1988	160	sector
SWITZERLAND	01/01/1992	146	sector
BELGIUM	02/01/1984	90	sector
FRANCE	01/01/1993	247	sector
GERMANY	01/01/1990	235	sector
ITALY	01/01/1986	150	sector
SINGAPORE	03/01/2005	100	sector
NORWAY	01/01/1990	50	supersector
IRELAND	01/01/1991	37	supersector
SWEDEN	01/01/1991	62	supersector
FINLAND	01/01/1996	47	supersector
AUSTRIA	01/01/1992	49	supersector
PORTUGAL	01/01/1993	48	supersector
DENMARK	01/01/1993	47	supersector
NEW ZEALAND	03/01/2000	50	supersector
ISRAEL	03/01/2000	49	supersector

Portfolio construction

Four levels of hierarchy :

- M_t: number of geographical **regions**
- $M_t^{j_1}$: number of **countries** in j_1-th region
- $M_t^{j_1,j_2}$: number of **industrial groupings** in j_2-th country of j_1-th region
- $M_t^{j_1,j_2,j_3}$: number of **stocks** in j_3-th industrial grouping of j_2-th country of j_1-th region

Number of stocks:

$$N_t = \sum_{j_1=1}^{M_t} \sum_{j_2=1}^{M_t^{j_1}} \sum_{j_3=1}^{M_t^{j_1,j_2}} M_t^{j_1,j_2,j_3}$$

- S_t^j: cum-dividend **price of** j**-th stock** (US dollars), $j = (j_1, j_2, j_3, j_4)$
- π_t^j: **portfolio weight** for j-th stock
- $\pi_t = (\pi_t^1, \pi_t^2, \ldots, \pi_t^{N_t})^\top$: **weights** of portfolio V_t^π

Portfolio value:

$$V_{t_i}^{\pi} = V_{t_{i-1}}^{\pi} \left(1 + \sum_{j=1}^{N_{t_{i-1}}} \pi_{t_{i-1}}^{j} \frac{S_{t_i}^{j} - S_{t_{i-1}}^{j}}{S_{t_{i-1}}^{j}} \right)$$

Rebalancing frequency: quarterly

HWI weights

Weight for j-th stock, with $j = (j_1, j_2, j_3, j_4)$:

$$\pi_t^{HWI,j} = \frac{1}{M_t} \frac{1}{M_t^{j_1}} \frac{1}{M_t^{j_1,j_2}} \frac{1}{M_t^{j_1,j_2,j_3}}$$

for $t \geq 0$.

MCI and EWI weights

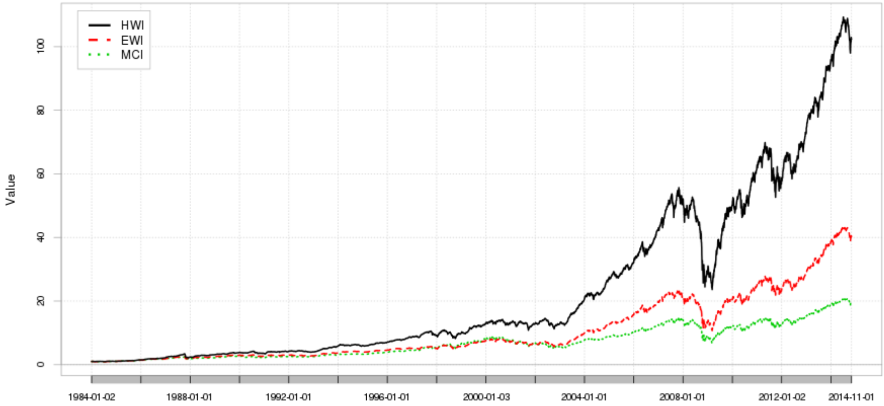
- Market Capitalization Weighted Index (MCI):

$$\pi_t^{MCI,j} = \frac{MV_t^j}{\sum_{k=1}^{N_t} MV_t^k}$$

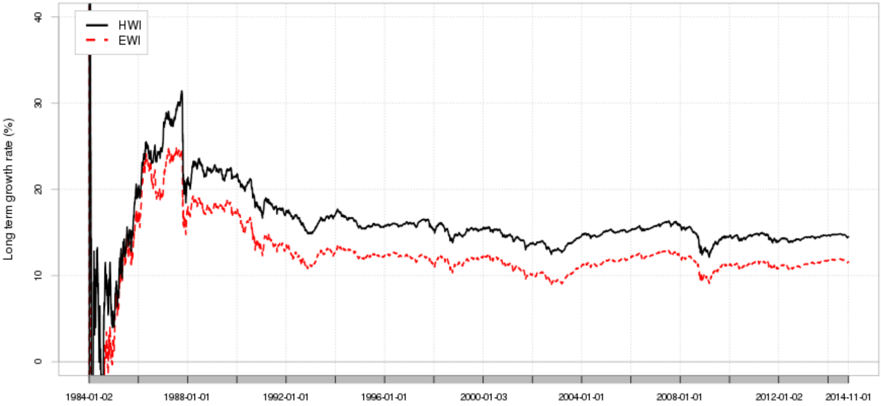
- Equal Weighted Index (EWI):

$$\pi_t^{EWI,j} = \frac{1}{N_t}$$

HWI, EWI and MCI



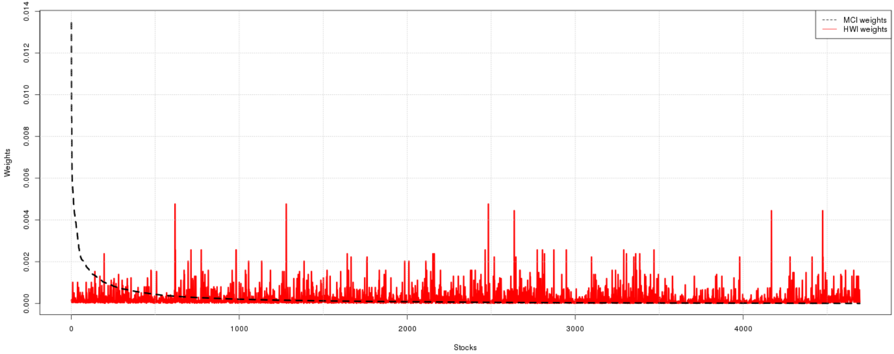
Long term-growth rates for HWI and EWI



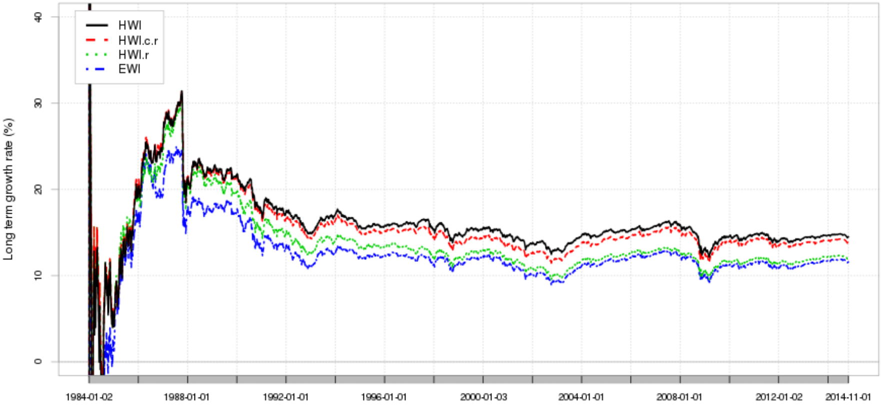
Supersector and country weights

ICB Supersector	HWI	EWI	MCI	Country	HWI	EWI	MCI
Industrial Goods & Services	**13.95**	**16.19**	**11.42**	CANADA	**16.67**	5.219	4.243
Personal & Household Goods	**6.889**	4.729	5.272	UNITED STATES	**16.67**	**21.26**	**48.51**
Real Estate	**6.586**	**8.649**	3.859	HONG KONG	**6.667**	2.706	4.499
Technology	6.051	6.966	**9.503**	JAPAN	**6.667**	**21.28**	**9.917**
Retail	5.932	5.944	5.492	AUSTRALIA	**6.667**	3.409	2.819
Basic Resources	5.917	3.622	2.244	SINGAPORE	**6.667**	2.13	1.27
Oil & Gas	5.482	6.221	8.978	NEW ZEALAND	**6.667**	1.065	0.1381
Food & Beverage	5.418	4.325	4.587	UNITED KINGDOM	2.083	**11.48**	**7.865**
Financial Services	5.146	**7.861**	4.23	SPAIN	2.083	2.45	1.807
Insurance	5.033	2.897	4.137	NETHERLANDS	2.083	2.28	1.238
Health Care	5.01	6.242	**10.02**	SWITZERLAND	2.083	3.11	3.293
Utilities	4.662	3.409	3.685	BELGIUM	2.083	1.917	0.8247
Telecommunications	4.491	1.513	4.183	FRANCE	2.083	5.262	4.531
Travel & Leisure	4.393	4.325	3.029	GERMANY	2.083	4.985	3.753
Construction & Materials	3.624	4.154	1.434	ITALY	2.083	3.174	1.397
Media	3.237	2.983	2.6	NORWAY	2.083	1.065	0.6474
Chemicals	3.107	2.812	2.699	IRELAND	2.083	0.7882	0.1599
Banks	2.956	4.708	9.696	SWEDEN	2.083	1.321	1.261
Automobiles & Parts	2.111	2.45	2.934	FINLAND	2.083	1.001	0.4259
				AUSTRIA	2.083	1.044	0.2295
				PORTUGAL	2.083	1.001	0.1515
				DENMARK	2.083	1.001	0.6733
				ISRAEL	2.083	1.044	0.3431

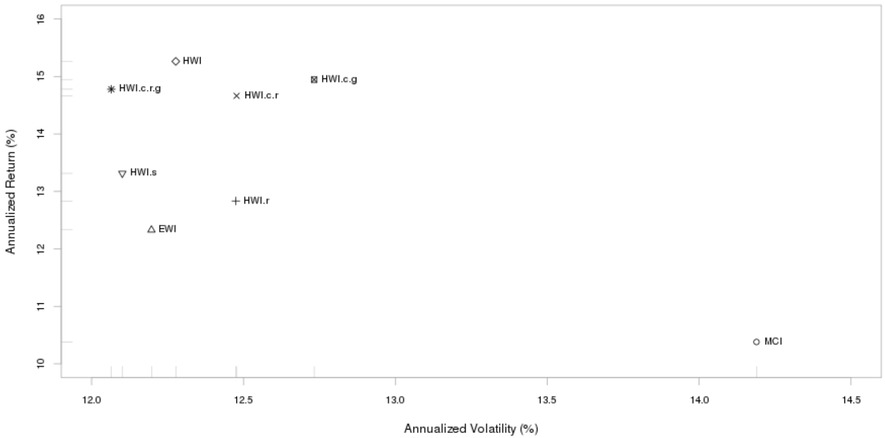
HWI and MCI weights



Other hierarchical groupings



Risk/Return characteristics



GR and Sharpe ratio

Index	GR	Average Return	Risk Premium	Volatility	Sharpe Ratio
MSCI	9.23	10.48	6.26	15.79	0.3963
MCI	9.37	10.38	6.16	14.19	0.4343
EWI	11.58	12.33	8.11	12.20	0.6650
HWI	14.50	15.26	11.05	12.28	0.8997

- MCI poorly positioned on mean-variance efficient frontier
- contradicts classical theory, Markowitz (1959) and Sharpe (1964)

VaR and ES

Index	VaR (95%)	ES (95%)
MCI	-0.012834	-0.021029
EWI	-0.011252	-0.018382
HWI	-0.010636	-0.018680

Drawdown

Index	Av. Drawdown	Av. Recovery
MCI	0.0199	14.7533
EWI	0.0187	11.9556
HWI	0.0169	9.4541

Kardaras & Platen (2010) $\rightarrow$ GP requires the shortest expected time to reach a target level

Transaction cost

Index	GR	Average Return	Risk Premium	Volatility	Sharpe Ratio
MCI-TC	9.20	10.22	6.00	14.19	0.4228
EWI-TC	11.30	12.05	7.83	12.20	0.6423
HWI-TC	14.15	14.92	10.70	12.28	0.8714

Testing the EMP

μ: "true" mean of all daily returns of all benchmarked stocks

$$H_0 : \mu \leq 0 \qquad \text{vs} \qquad H_1 : \mu > 0$$

Test for mean daily annualized percentage return of all benchmarked stocks

Benchmark	Sample mean	Standard Error	99% LCI	99% UCI	Z-test	p-value
MCI-TC	3.504079	0.142278	3.137594	3.870563	24.628	0
EWI-TC	0.936921	0.141376	0.572761	1.301081	6.627	0
HWI-TC	-1.671584	0.141828	-2.036909	-1.306259	-11.786	1
HWI.c.r-TC	-1.072318	0.141815	-1.437608	-0.707027	-7.561	1
HWI.c.g-TC	-1.238168	0.141928	-1.603750	-0.872587	-8.724	1
HWI.c.r.g-TC	-1.364769	0.141681	-1.729714	-0.999823	-9.633	1

Bootstrap test

Benchmark	Bootstrap mean	99% LCI	99% UCI	Test statistic	p-value
MCI-TC	3.502956	3.140459	3.910323	23.133	0
EWI-TC	0.936228	0.571123	1.398584	6.256	0
HWI-TC	-1.664403	-2.018709	-1.277940	-11.529	1
HWI.c.r-TC	-1.074673	-1.419532	-0.674604	-7.392	1
HWI.c.g-TC	-1.243531	-1.617380	-0.893065	-8.177	1
HWI.c.r.g-TC	-1.361862	-1.725689	-1.001110	-8.968	1

Test for mean daily returns of HWI-TC benchmarked portfolios

Benchmark	Sample mean	99 % LCI	99 % UCI	Z-test	p-value
MCI-TC	-5.239304	-8.707531	-1.771077	-3.89	1
EWI-TC	-2.943474	-4.914147	-0.972799	-3.85	1
HWI.c.r-TC	-0.587549	-1.219394	0.044296	-2.4	0.99
HWI.c.g-TC	-0.440414	-2.217511	1.336683	-0.64	0.74
HWI.c.r.g-TC	-0.477921	-1.354879	0.399037	-1.4	0.92

Stylized Hierarchical Market Model (HMM)

- **realistic** stylized hierarchical stock market model
- **HWI coincides exactly with GP**
- exploits **hierarchical stock market structure**
- sets market price of risk processes equal
- sets risk aversion processes equal

Stock Price

- in mean-variance optimization j-th company applies **risk aversion** γ_t to invest fraction $\frac{1}{\gamma_t}$ in its **'own' GP**, $S_t^{j,GP}$
- fraction $1 - \frac{1}{\gamma_t}$ in **risk-free asset**; see e.g., Campbell & Viceira (2002)

$$\frac{dS_t^j}{S_t^j} = \frac{1}{\gamma_t} \frac{dS_t^{j,GP}}{S_t^{j,GP}} + \left(1 - \frac{1}{\gamma_t}\right) r_t dt$$

GP of jth company

Filipović & Platen (2009) $\rightarrow$ GP of jth company:

$$
\begin{aligned}
\frac{dS_t^{j,GP}}{S_t^{j,GP}} &= r_t dt + \theta_t(\theta_t dt + dW_t^{j_1}) + \theta_t(\theta_t dt + dW_t^{j_1,j_2}) \\
&+ \theta_t(\theta_t dt + dW_t^{j_1,j_2,j_3}) + \theta_t(\theta_t dt + dW_t^{j_1,j_2,j_3,j_4})
\end{aligned}
$$

Driven by the uncertainties:

- W^{j_1,j_2,j_3,j_4}: company specific
- W^{j_1,j_2,j_3}: industrial grouping
- W^{j_1,j_2}: country
- W^{j_1}: region

Stock Value

j-th cum-dividend stock, $j = (j_1, j_2, j_3, j_4)$:

$$\frac{dS_t^j}{S_t^j} = r_t dt + \frac{1}{\gamma_t}\Big(\theta_t(\theta_t dt + dW_t^{j_1}) + \theta_t(\theta_t dt + dW_t^{j_1,j_2}))$$
$$+ \quad \theta_t(\theta_t dt + dW_t^{j_1,j_2,j_3}) + \theta_t(\theta_t dt + dW^{j_1,j_2,j_3,j_4})\Big)$$

HWI under HMM

$$
\begin{aligned}
\frac{dS_t^{HWI}}{S_t^{HWI}} &= r_t dt + \frac{1}{\gamma_t} \frac{1}{M_t} \sum_{j_1=1}^{M_t} \Bigg(\theta_t (\theta_t dt + dW_t^{j_1}) \\
&+ \frac{1}{M_t^{j_1}} \sum_{j_2=1}^{M_t^{j_1}} \Bigg(\theta_t (\theta_t dt + dW_t^{j_1,j_2}) \\
&+ \frac{1}{M_t^{j_1,j_2}} \sum_{j_3=1}^{M_t^{j_1,j_2}} \Bigg(\theta_t (\theta_t dt + dW_t^{j_1,j_2,j_3}) \\
&+ \frac{1}{M_t^{j_1,j_2,j_3}} \sum_{j_4=1}^{M_t^{j_1,j_2,j_3}} \theta_t (\theta_t dt + dW_t^{j_1,j_2,j_3,j_4}) \Bigg) \Bigg) \Bigg)
\end{aligned}
$$

Benchmarked jth stock

$$
\begin{aligned}
\frac{d\hat{S}_t^j}{\hat{S}_t^j} &= \sum_{k_1=1}^{M_t} (\psi_t^{j,k_1} dW_t^{k_1} + \sum_{k_2=1}^{M_t^{k_1}} (\psi_t^{j,k_1,k_2} dW_t^{k_1,k_2} \\
&+ \sum_{k_3=1}^{M_t^{k_1,k_2}} (\psi_t^{j,k_1,k_2,k_3} dW_t^{k_1,k_2,k_3} + \sum_{k_4=1}^{M_t^{k_1,k_2,k_3}} \psi_t^{j,k_1,k_2,k_3,k_4} dW_t^{k_1,k_2,k_3,k_4})))
\end{aligned}
$$

with

$$
\psi_t^{j,k_1,\ldots,k_n} = \begin{cases} \frac{1}{\gamma_t}\theta_t \left(1 - \frac{1}{M_t^{j_1} \ldots M_t^{j_1,\ldots,j_{n-1}}}\right) & \text{for } k_i = j_i \text{ for all } i \in \{1,\ldots,n\} \\ -\frac{1}{\gamma_t}\theta_t \frac{1}{M_t^{j_1} \ldots M_t^{j_1,\ldots,j_{n-1}}} & \text{otherwise} \end{cases}
$$

Since benchmarked stocks are **local martingales**, according to Theorems 3.1. and 4.1 in Filipović & Platen (2009), the **HWI is the GP**.

General jth benchmarked stock price

$$\frac{d\hat{S}_t^j}{\hat{S}_t^j} = \sum_{k_1=1}^{\bar{K}M} (\psi_t^{j,k_1} dW_t^{k_1} + \sum_{k_2=1}^{\bar{K}M} (\psi_t^{j,k_1,k_2} dW_t^{k_1,k_2}$$

$$+ \cdots + \sum_{k_H=1}^{\bar{K}M} \psi_t^{j,k_1,k_2,\ldots,k_H} dW_t^{k_1,k_2,\ldots,k_H})),$$

$j = (j_1, j_2, \ldots, j_H) \in \Gamma_M = (1, 2, \ldots, \bar{K}M)^H$

Return process of benchmarked portfolio

$$
\begin{aligned}
d\hat{Q}_t^{\pi_M} &= \frac{d\hat{V}_t^{\pi_M}}{\hat{V}_t^{\pi_M}} = \sum_{j\in\Gamma_M} \pi_{M,t}^j \frac{d\hat{S}_t^j}{\hat{S}_t^j} \\
&= \sum_{k_1=1}^{\bar{K}M} \sum_{j\in\Gamma_M} \pi_{M,t}^j \psi_t^{j,k_1} dW_t^{k_1} \\
&+ \sum_{k_1=1}^{\bar{K}M} \sum_{k_2=1}^{\bar{K}M} \sum_{j\in\Gamma_M} \pi_{M,t}^j \psi_t^{j,k_1,k_2} dW_t^{k_1,k_2} \\
&+ \cdots + \sum_{k_1=1}^{\bar{K}M} \sum_{k_2=1}^{\bar{K}M} \cdots \sum_{k_H=1}^{\bar{K}M} \sum_{j\in\Gamma_M} \pi_{M,t}^j \psi_t^{j,k_1,k_2,\ldots,k_H} dW_t^{k_1,k_2,\ldots,k_H}
\end{aligned}
$$

Definition: Sequence of benchmarked approximate GP processes

A sequence of benchmarked portfolio processes, each with return process $\hat{Q}^{\pi_M}$, is a sequence of benchmarked approximate GP processes if for all $\varepsilon > 0$ and $t \in [0, \infty)$

$$\lim_{M \to \infty} P\left(\frac{d[\hat{Q}^{\pi_M}_{\cdot}]_t}{dt} > \varepsilon \right) = 0$$

.

Assumption

For given $k_1, k_2, \dots, k_h \in \{1, 2, \dots, \bar{K}M\}$ assume for all $M \in \{2, 3, \dots\}$ and all $h \in \{1, 2, \dots, H\}$

$$\sum_{j \in \Gamma_M} |\psi_t^{j,k_1,k_2,\dots,k_h}| \le (\bar{K}M)^{H-h} \sigma_t,$$

where

$$E((\sigma_t)^2) \le \bar{\sigma}^2 < \infty$$

for all $t \in [0, \infty)$.

Diversification Theorem

A sequence of benchmarked portfolios $(\hat{V}^{\pi_M})_{M\in\{2,3,\dots\}}$, is a sequence of benchmarked approximate GPs, if for each $M \in \{2, 3, \dots\}$

$$\max_{j\in\Gamma_M} |\pi^j_{M,t}| \leq C M^{\xi - H}$$

for $\xi \in [0, \frac{1}{2})$, $C \in (0, \infty)$, and $t \in [0, \infty)$.

Proof of the Diversification Theorem

$$
\begin{aligned}
\frac{d[\hat{Q}_{\cdot}^{\pi_M}]_t}{dt} &= \sum_{k_1=1}^{\bar{K}M} \Big(\sum_{j \in \Gamma_M} \pi_{M,t}^{j} \psi_t^{j,k_1} \Big)^2 \\
&+ \sum_{k_1=1}^{\bar{K}M} \sum_{k_2=1}^{\bar{K}M} \Big(\sum_{j \in \Gamma_M} \pi_{M,t}^{j} \psi_t^{j,k_1,k_2} \Big)^2 \\
&+ \cdots + \sum_{k_1=1}^{\bar{K}M} \sum_{k_2=1}^{\bar{K}M} \cdots \sum_{k_H=1}^{\bar{K}M} \Big(\sum_{j \in \Gamma_M} \pi_{M,t}^{j} \psi_t^{j,k_1,k_2,\ldots,k_H} \Big)^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{d[\hat{Q}^{\pi M}_{\cdot}]_t}{dt} &\leq (\max_{j\in\Gamma_M} |\pi^j_{M,t}|)^2 \Bigg(\sum_{k_1=1}^{\bar{K}M} (\sum_{j\in\Gamma_M} |\psi_t^{j,k_1}|)^2 \\
&+ \sum_{k_1=1}^{\bar{K}M} \sum_{k_2=1}^{\bar{K}M} (\sum_{j\in\Gamma_M} |\psi_t^{j,k_1,k_2}|)^2 \\
&+ \cdots + \sum_{k_1=1}^{\bar{K}M} \sum_{k_2=1}^{\bar{K}M} \cdots \sum_{k_H=1}^{\bar{K}M} (\sum_{j\in\Gamma_M} |\psi_t^{j,k_1,k_2,\ldots,k_H}|)^2 \Bigg) \\
&\leq (\max_{j\in\Gamma_M} |\pi^j_{M,t}|)^2 \sigma_t^2 \sum_{h=1}^{H} (\bar{K}M)^h (\bar{K}M^{H-h})^2 \\
&\leq (\max_{j\in\Gamma_M} |\pi^j_{M,t}|)^2 \sigma_t^2 \bar{K}^{2H} \sum_{h=1}^{H} (M)^h (M^{H-h})^2 \\
&= \sigma_t^2 (\max_{j\in\Gamma_M} |\pi^j_{M,t}|)^2 \bar{K}^{2H} M^{2H-1} \sum_{h=1}^{H} \left(\frac{1}{M}\right)^{h-1} \\
&\leq \sigma_t^2 \bar{K}^{2H} (\max_{j\in\Gamma_M} |\pi^j_{M,t}|)^2 \frac{M^{2H-1}}{1-\frac{1}{M}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d[\hat{Q}^{\pi_M}_{\cdot}]_t}{dt} &\leq C^2 \sigma_t^2 \bar{K}^{2H-h} M^{2(\xi-H)} M^{2H-1} (1 - M^{-1})^{-1} \\
&\leq C^2 \sigma_t^2 \bar{K}^{2H-h} M^{2\xi-1} (1 - M^{-1})^{-1} \\
&\leq 2C^2 \sigma_t^2 \bar{K}^{2H-h} M^{2\xi-1}
\end{aligned}
$$

Since $\xi \in [0, \frac{1}{2})$, it follows by the Markov inequality for each $\varepsilon > 0$ and $t \in [0, T]$ that

$$\begin{aligned} \lim_{M\to\infty} P\left(\frac{d[\hat{Q}_{\cdot}^{\pi_M}]_t}{dt} > \varepsilon\right) &\leq \lim_{M\to\infty} \frac{1}{\varepsilon} E\left(\frac{d[\hat{Q}_{\cdot}^{\pi_M}]}{dt}\right) \\ &\leq \frac{2C^2}{\varepsilon} \bar{\sigma}^2 \bar{K}^{2H} \lim_{M\to\infty} M^{2\xi-1} = 0 \end{aligned}$$

Conclusions

- deep **connection between EMP and GP**
- **theoretically optimal stock portfolios cannot be implemented accurately** enough to be useful
- GP approximated by **HWI**, which does not rely on estimation

- **EMP is difficult to reject empirically** when using HWI as proxy of GP
- HWI useful in practical **portfolio optimization and risk management**
- HWI plays central role in benchmark pricing theory