Heterogeneity in mortality: from seminal contributions to recent actuarial applications

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Agenda

1. Introduction
2. Heterogeneity: the awareness
3. Heterogeneity: formal approaches
4. Seminal contributions
5. Some recent contributions
6. Heterogeneity in actuarial evaluations
7. Concluding remarks

1 INTRODUCTION

MOTIVATION

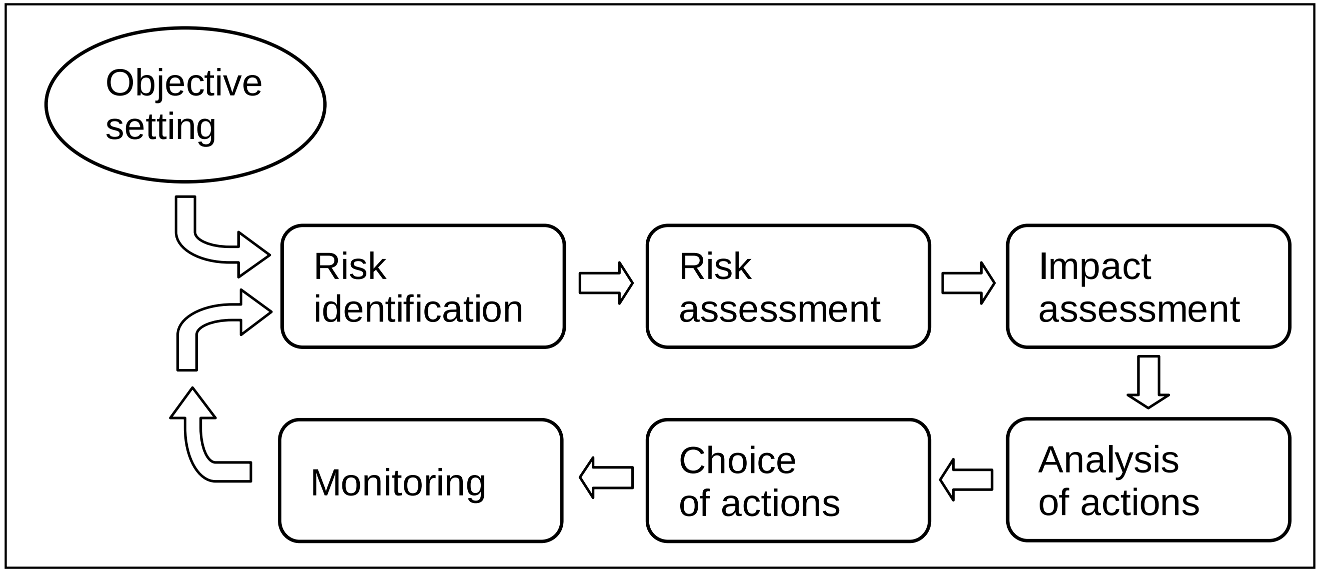
A huge number of scientific and technical contributions in the field of heterogeneity in respect of mortality, in particular:

- in demography
  - ▷ e.g., to explain mortality deceleration at high ages in terms of unobservable heterogeneity factors (a controversial issue !)
- in actuarial science
  - ▷ to represent the impact of observable heterogeneity factors on individual mortality $\Rightarrow$ appropriate pricing and reserving models
  - ▷ to assess the impact of unobservable heterogeneity factors on the risk profile of insurance and annuity portfolios $\Rightarrow$ solvency requirements, capital allocation

Aim of this presentation: to provide some guidelines, hopefully useful in exploring the complex network of contributions to the analysis of mortality heterogeneity

We first focus on some forerunners in the demographic and actuarial fields, then moving to more recent contributions, with actuarial applications as our ultimate target

Given the ultimate target, we follow the guidelines provided by the logical structure of the Risk Management (RM) process



The RM process (1)

EXPLORING HETEROGENEITY: SOME CLASSIFICATIONS

Awareness of heterogeneity: a first result of the *risk identification* phase

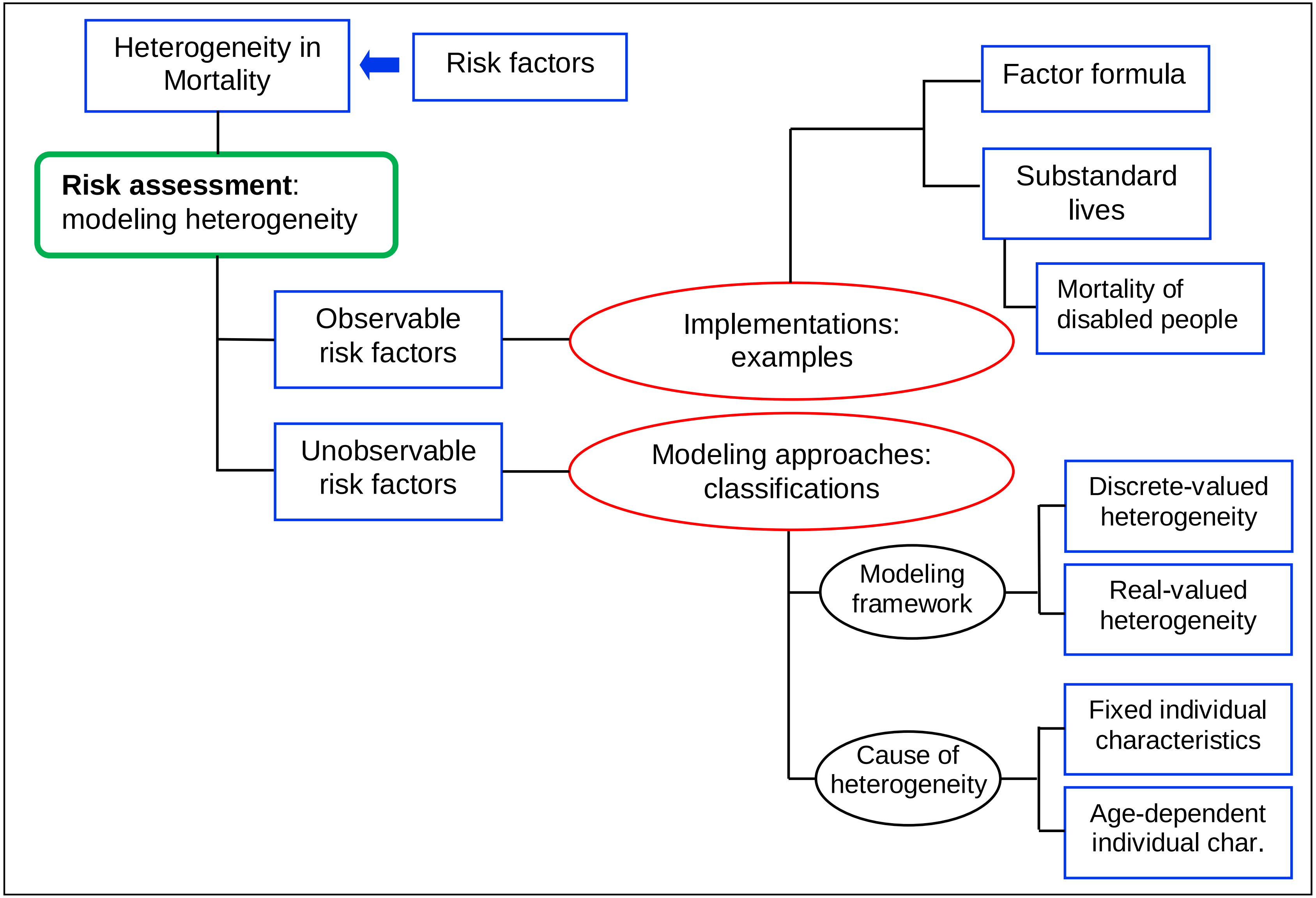
Second result: heterogeneity due to

- observable risk factors
- unobservable risk factors

Different approaches adopted in the *risk assessment* phase

Classification particularly relevant in actuarial applications

See following figure

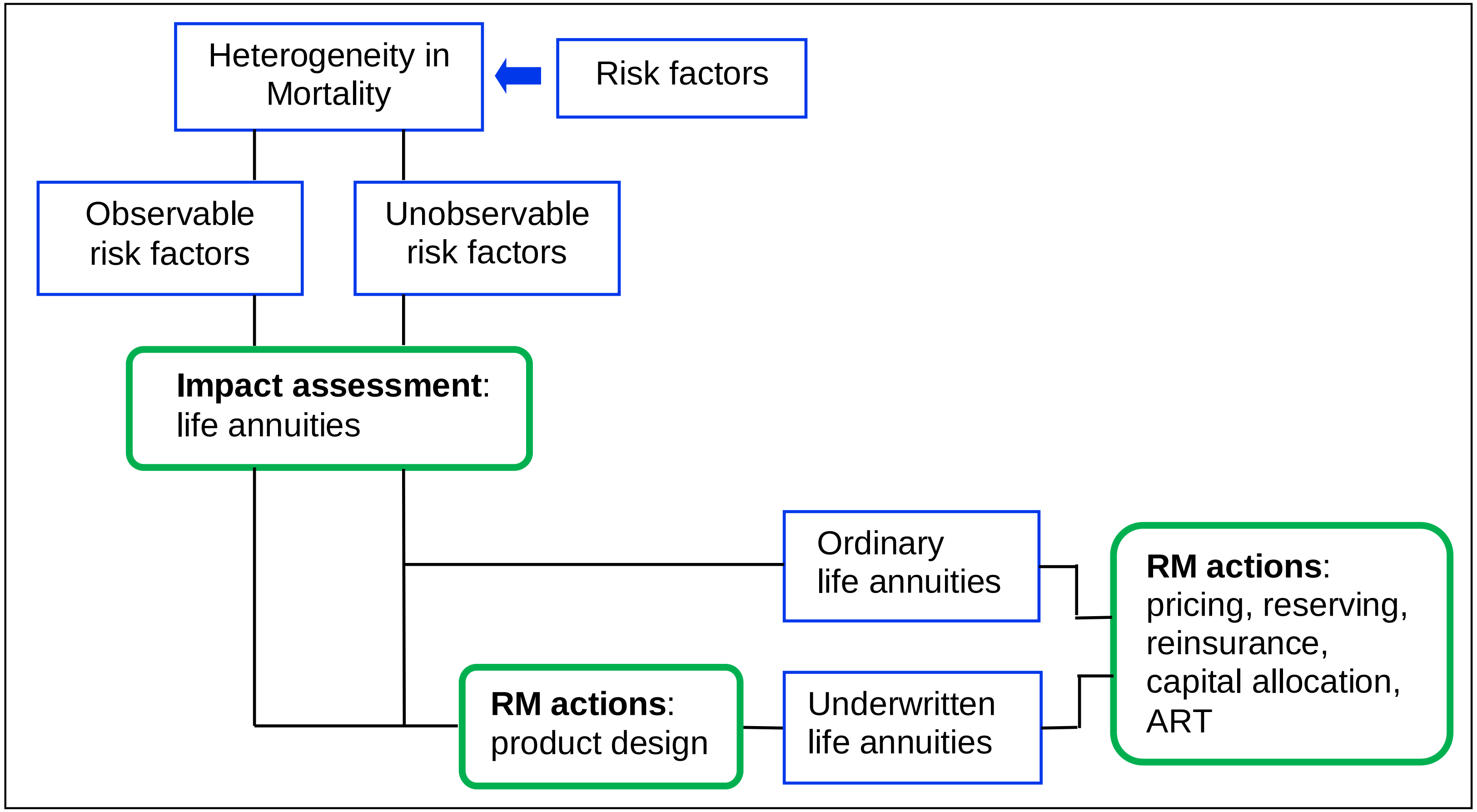


The biometric side: risk assessment

Impact assessment phase: quantifying the impact on relevant results (profits, NAV, etc.) $\Rightarrow$ the preliminary step to compare and then choose *RM actions*, among which:

- Product design and related pricing and reserving
- Risk hedging actions
  - ▷ reinsurance
  - ▷ capital allocation
  - ▷ ART (mortality-linked securities, swaps, . . .)

See folllowing Figure



The actuarial side: impact assessment and risk management actions

2 HETEROGENEITY: THE AWARENESS

Presence of heterogeneity in respect of mortality: intuitive and supported by statistical evidence, for example:

- males vs females
- environment
- ...

$\Rightarrow$ Heterogeneity recognized in early contributions in demography and actuarial science (see below)

Recently, strengthened interest:

- in demography (and actuarial science) $\Rightarrow$ link between heterogeneity due to non-observable risk factors and mortality deceleration at high ages (a controversial issue; see for example Pitacco [2016b] and references therein)
- in actuarial science
  - $\Rightarrow$ design of underwritten (or “special-rate”) annuity products
  - $\Rightarrow$ assessment of portfolio risk profile in presence of heterogeneity

Among the antecedents: Francis Corbaux

Corbaux [1833]: "*The object of consideration . . . various classes susceptible of being discriminated amongst any extensive population, . . .*"

Splitting the population into groups $\Rightarrow$ 12 risk factors (or proxies) involved, concerning:

- ▷ health
- ▷ lifestyle
- ▷ environment

$\Rightarrow$ 5 groups

Remark

Scheme | risk factors $\Rightarrow$ rating classes | adopted in defining rating criteria for underwritten (or special-rate) annuities; see e.g. Rinke [2002]

Some forerunners in the Fifties

- Claudio de Ferra [1954]
  - ▷ heterogeneous population split in a given number of homogeneous groups
  - ▷ age-pattern of mortality described by a Makeham law for each group, with specific parameters
  - ▷ population structure described by the distribution of the parameters
  - ▷ problem: find the law describing the mortality in the heterogeneous population, given the distribution of the Makeham parameters
  - ▷ particular application: insured population consisting of normal risks and substandard risks, with different extra-mortality levels

- Louis Levinson [1959]
  - ▷ heterogeneous population split into a given number of homogeneous *strata* (same probability of dying)
  - ▷ each individual can move from one stratum to another one $\Rightarrow$ probably, the first attempt of modeling heterogeneity in a *dynamic setting* (see the following)
- Robert Eric Beard [1959]
  - ▷ a seminal contribution to modeling the heterogeneity due to non-observable risk factors
  - ▷ individual mortality described by a Makeham law, with parameters depending on a specific *longevity factor*
  - ▷ distribution of the longevity factor in the population described by a gamma distribution
  - ▷ $\Rightarrow$ mortality in the population described by a Perks law (logistic model $\Rightarrow$ deceleration of mortality at high ages)
  - ▷ starting point of the *frailty theory* (see the following)

3 HETEROGENEITY: FORMAL APPROACHES

Looking at demographical and actuarial literature, we can recognize two basic approaches to heterogeneity in mortality

AN INTUITIVE APPROACH

A heterogeneous population can be considered as a (finite) set of (more or less) homogeneous groups

$\Rightarrow$ The population life table can be represented as a (finite) mixture of life tables, each one representing the age-pattern of mortality in the relevant group

Formally: refer to a biometric function f; for example:

- hazard function (instantaneous force of mortality) μ
- annual probabilities of dying q
- life table function ℓ
- life expectancy e

For a population split into n groups:

$$f = w_1\, f^{(1)} + w_2\, f^{(2)} + \cdots + w_n\, f^{(n)} \tag{1}$$

Several specific models in the framework of Eq. (1)

For example:

- functions $f^{(1)}, f^{(2)}, \ldots, f^{(n)}$
  - ▹ can be suggested by various risk factors (e.g. individual health status, individual occupation, geographical area, etc.)
  - ▹ known vs unknown
- weights $w_1, w_2, \ldots, w_n$ known vs unknown
- individual age-pattern of mortality over time
  - ▹ fixed $\Leftrightarrow$ the individual remains in a given group
  - ▹ variable $\Leftrightarrow$ the individual can change group

Eq. (1): an example of discrete (finite) approach to heterogeneity

PARAMETRIC REPRESENTATION: A (RATHER) GENERAL SETTING

Looking for a general setting, suggested by models developed in demography and actuarial sciences

- Choose a biometric function f to represent the age-pattern of mortality in a group (general population in a country, pension fund, insurance portfolio, etc.)
- Express specific age-pattern of mortality, e.g.:
  - ▷ individual mortality
  - ▷ mortality in a subgroup

  as a transform of f, involving various parameters

Examples of function f: see above (μ, q, $\dots$)

In terms of the hazard function:

$$\mu_{x,t}^{[\text{spec}]} = \Phi\big[\mu_{x+s+t}; \rho_{x,t}^{(1)}, \dots, \rho_{x,t}^{(r)}; z_{x,t}\big] \quad (2)$$

where:

- x = given age (e.g. $x = 0$, or $x =$ age at policy issue)
- t = past duration ($\Rightarrow\ x + t =$ current age)
- μ_{x+s+t} = "standard" hazard function at age $x + s + t$
- s = age-shift parameter, summarizing the impact of some observable risk factors ($s \geq 0$)
- $\rho_{x,t}^{(j)}$ = impact of observable risk factor j, $j = 1, \dots, r$
- $z_{x,t}$ = overall impact of non-observable risk factors

Parametric representation: examples

Observable risk factors (disregarding non-observable risk factors)

(See, for example: Pitacco [2012])

1. Mortality of *substandard lives* (application: term insurance rating, underwritten life annuities rating); see e.g. Ainslie [2000]

$$\mu_{x,t}^{(h)} = A_t^{(h)} \, \mu_{x+s^{(h)}+t} + B_t^{(h)} \tag{3}$$

   where:

   - $A_t^{(h)}, B_t^{(h)}, s^{(h)}$ summarize the impact of observable risk factors $\rho_{x,t}^{(1)}, \dots, \rho_{x,t}^{(r)}$ via the substandard category h
   - past duration effect allowed for via the parameters $A_t^{(h)}, B_t^{(h)}$

In particular, disregarding past duration effect:

1(a) Linear model

$$\mu_{x+t}^{(h)} = A^{(h)} \, \mu_{x+t} + B^{(h)} \tag{4}$$

- ▷ multiplicative model

$$\mu_{x+t}^{(h)} = A^{(h)} \, \mu_{x+t} \tag{5}$$

- ▷ additive model

$$\mu_{x+t}^{(h)} = \mu_{x+t} + B^{(h)} \tag{6}$$

1(b) Age-shift model

$$\mu_{x+t}^{(h)} = \mu_{x+s^{(h)}+t} \tag{7}$$

2. Mortality of disabled people - Formulae with additive extra-mortality, applied to the q's (instead of μ), proposed by Rickayzen and Walsh [2002] for LTC insurance; see also Rickayzen [2007], and the sensitivity analysis in Pitacco [2016a]

$$q_{x+t}^{(h)} = q_{x+t} + \Delta(x+t; \alpha, h) \tag{8}$$

with:

$$\Delta(x+t; \alpha, h) = \frac{\alpha}{1 + 1.1^{50-(x+t)}} \, \frac{\max\{h-5, 0\}}{5} \tag{9}$$

where:

- parameter h expresses the LTC severity category summarizing the risk factors $\rho_{x,t}^{(1)}, \dots, \rho_{x,t}^{(r)}$, according to the UK OPCS scale; in particular:
  - ▷ $0 \le h \le 5$ denotes less severe LTC states, with no significant impact on mortality
  - ▷ $6 \le h \le 10$ denotes more severe LTC states, implying an extra-mortality
- parameter α chosen according to the type of the standard mortality q (e.g. population mortality vs insured lives mortality)

3. Numerical rating system (Factor formula): New York Life Insurance, 1919

$$q_{x+t}^{[\text{spec}]} = q_{x+t} \underbrace{\left(1 + \sum_{k=1}^{r} \rho^{(k)}\right)}_{A^{(h)} \gtreqless 0} \qquad (10)$$

An example of the multiplicative model, applied to the q's (instead of μ), not restricted to substandard lives

For details, see: Cummins et al. [1983]

Non-observable risk factors

Assume, for simplicity of notation, that μ (or q, etc.) refers to a group of individuals which is homogeneous in respect of the observable risk factors

Then Eq. (2) reduces to:

$$\mu_{x,t}^{[\text{spec}]} = \Phi\big[\mu_{x+t}; z_{x,t}\big] \tag{11}$$

Eq. (11) encompasses a number of models; in particular, as regards the impact of non-observable risk factors on individual mortality:

- ▷ $z_{x,t} = z_x$, i.e. independent of the attained age $\Rightarrow$ impact does not change throughout time (see: *Fixed-frailty models*)
- ▷ $z_{x,t}$ depends on the attained age $x + t$ $\Rightarrow$ impact varies throughout time (see: *Dynamic settings*)

4 SEMINAL CONTRIBUTIONS

THE DISCRETE APPROACH

Analyzing one-year mortality

A. H. Pollard [1970]: *The population value of q_x may be considered as the weighted sum of the rates of mortality of groups of persons suffering from particular disabilities. If there is any variation in the proportion suffering, for example, from particular heart conditions or if there is any variation in the degree of such impairments then variations in the population value of q_x must be expected in addition to the random variations which occur in the observed rate of mortality when q_x is constant.*

Two aspects in particular emerge:

- population split into (homogeneous) groups
- (possible) dynamic features
  - ▷ relative size of the groups
  - ▷ impact of risk factors on mortality within some groups

Various settings considered by Pollard [1970]:

1. Population = 1 group, all individuals with given $q_x \Rightarrow$ binomial distribution of the number of deaths
2. Population split into r groups, each one with given size $n^{(i)}$ and given probability $q_x^{(i)}$ ("known" heterogeneity) $\Rightarrow$ variance of the total number of death lower than in case 1
3. Population = 1 group, all individuals with random q_x, with given expected value and variance $\Rightarrow$ variance of the total number of death higher than in case 1
4. Combining 2 and 3 $\Rightarrow$ variance of the total number of death, compared to case 1, lowered because 2 and increased because 3
5. Population split into r groups, each one with random size $n^{(i)}$ but given probability $q_x^{(i)}$ ("random" heterogeneity) $\Rightarrow$ variance of the total number of death, compared to case 1, lowered because splitting into groups and increased because random sizes

6. Population split into r groups, each one with random size $n^{(i)}$ and random probability $q_x^{(i)}$ ("random" heterogeneity) $\Rightarrow$ the general setting, including all the above cases

All the above cases can be traced back to the scheme defined by Eq. (1), referred to one-year mortality

Other discrete models

Redington [1969]:

- ▷ a heterogeneous population can be split into a given number of homogeneous sub-populations
- ▷ mortality in each sub-population is described by a Gompertz law with specific parameters
- ▷ symmetric distribution assumed for the two parameters
- ▷ average force of mortality in the population compared to the "central" force of mortality obtained by assigning to the parameters their modal values

Keyfitz and Littman [1979]:

- very simplified model, anyhow valuable because it marks some significant features of heterogeneity in mortality
- a heterogeneous population can be split into a given number of homogeneous sub-populations
- shares of sub-populations unknown because of unobservable heterogeneity
- focus on the impact of heterogeneity on the average expected lifetime
- conclusion: disregarding heterogeneity $\Rightarrow$ underestimation of the average expected lifetime
- important issue in the management of life annuity portfolios and pension plans

UNOBSERVABLE HETEROGENEITY: FIXED-FRAILTY MODELS

See: Beard [1959, 1971], Vaupel et al. [1979]

Assume that:

- the heterogeneity due to non-observable risk factors is expressed by the individual *frailty*
- the individual frailty (a positive real number) remains constant over the whole life span

For a person current age y with frailty level z ($z > 0$) $\Rightarrow$ (conditional) force of mortality denoted by $\mu_y(z)$

Probability density function (pdf) of the frailty distribution at age y: $g_y(z)$

Standard force of mortality (i.e. for $z = 1$):

$$\mu_y = \mu_y(1) \tag{12}$$

Average force of mortality in the cohort:

$$\bar{\mu}_y = \int_0^{+\infty} \mu_y(z)\, g_y(z)\, \mathrm{d}z \tag{13}$$

Specific models and results rely on:

1. relation between $\mu_y(z)$ and the standard force of mortality μ_y
2. pdf of the frailty distribution at a given age x, e.g. $x = 0$: $g_0(z)$
3. mortality law $\Rightarrow$ model for μ_y

In particular, combining:

1. Multiplicative model for the force of mortality:

$$\mu_y(z) = z\, \mu_y \tag{14}$$

   - ▷ a simple implementation of model (11), with $x + t = y$ and $z_{x,t} =$ const.
2. Gamma distribution with parameters δ, θ
3. Gompertz law $\mu_y = \alpha\, \mathrm{e}^{\beta\, y}$

then:

$$\bar{\mu}_y = \frac{\alpha' \, \mathrm{e}^{\beta y}}{\delta' \, \mathrm{e}^{\beta y} + 1} \tag{15}$$

Gompertz - Gamma model $\Rightarrow$ particular case of 1st Perks law (i.e. the Beard law), with parameters α', δ', depending on the parameters δ, θ of the frailty distribution

$\Rightarrow$ logistic hazard function

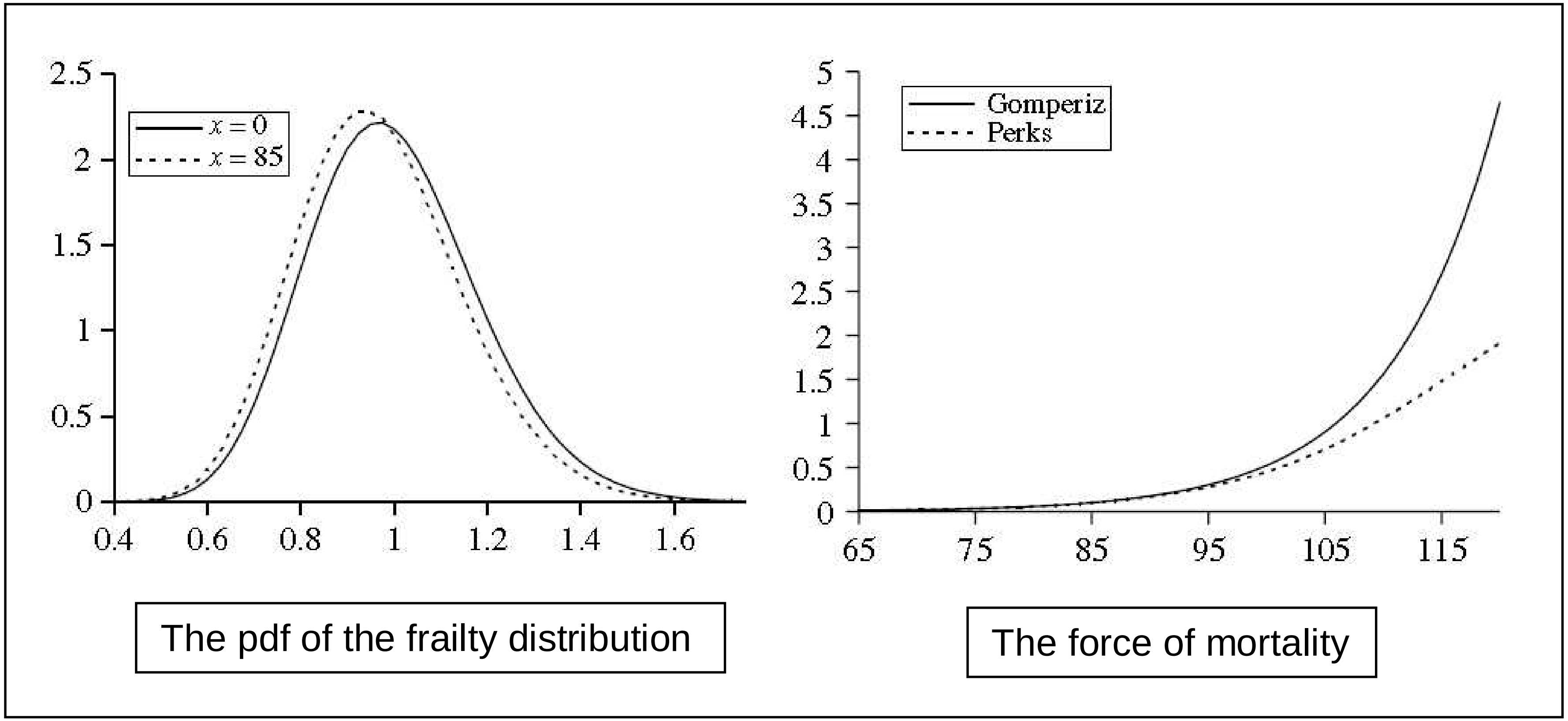
$\Rightarrow$ deceleration in cohort mortality implied by individual frailty

For a formal presentation see, for example: Haberman and Olivieri [2014], Pitacco et al. [2009]

Several generalizations proposed; for example:

- ▷ force of mortality expressed by Makeham law (Beard [1959]), Weibull law (Manton et al. [1986])
- ▷ frailty distribution given by inverse Gaussian (Manton et al. [1986], Butt and Haberman [2004]), shifted Gamma distribution (Martinelle [1987])

For a more general framework, see: Duchateau and Janssen [2008], Wienke [2003]



The Gompertz-Gamma model (Source: Pitacco et al. [2009])

UNOBSERVABLE HETEROGENEITY: DYNAMIC SETTINGS

Louis Levinson [1959]:

- Approach which in modern terms may be considered "multistate"
- Every population is heterogeneous in respect of mortality; even if split into classes (e.g. insureds accepted as "normal risks"), each class is heterogeneous $\Rightarrow$ homogeneous subgroups
- Definition of *mortality strata* $\Rightarrow$ each stratum consists of individuals with the same probability of death (regardless of age)
- Individuals move from one stratum to another one, in particular because of ageing, and in general because of *deterioration*
  - ▷ an example of *dynamic setting*
- Ultimate aim: construction of life tables allowing for strata; application to US life tables

Hervé Le Bras [1976]: individual frailty as a Markov process

- Framework: mortality modeling to define the limit age
- Basic idea: a mortality law must be the result of assumptions on the structure of a *process* describing the evolution of individual mortality throughout the whole life
- Assumptions:
  - ▷ each individual has an "initial frailty" (*faute*)
  - ▷ definition of "transition" probabilities: the probability of an increase in frailty (*nouvelle faute*) and the probability of death are proportional to the current frailty level (Markov hypothesis)
  - ▷ the resulting mortality law follows the Gompertz pattern up to some age, then tending to a limit ($\Rightarrow$ logistic-like shape)
  - ▷ we can recognize an implementation of model (11), in terms of q (instead of μ)

5 SOME RECENT CONTRIBUTIONS

THE MARKOV FRAMEWORK

Lin and Liu [2007]:

- A finite-state Markov process is adopted to model human mortality
- Individual health status is represented by the *physiological age*, and modeled by the Markov process
  - ▷ each Markov state represents an outcome of the physiological age
- Random time of death then follows a phase-type distribution
- Frailty
  - ▷ measured by the physiological age
  - ▷ distributed according to the distribution of individuals among age classes

Liu and Lin [2012]:

- Generalize the previous model by introducing uncertainty in mortality $\Rightarrow$ subordinated Markov model

SPLITTING A HETEROGENEOUS POPULATION INTO (HOMOGENEOUS) SUBPOPULATIONS

Avraam et al. [2013]:

- Assumption: age pattern of mortality described by the Gompertz law
- How to explain deviations from the Gompertz pattern, which can be observed in a population ?
  - heterogeneity explains in particular mortality plateau at high ages $\Rightarrow$ splitting into (homogeneous) subpopulations, each one with different Gompertz parameters
  - stochastic effects explain other deviations from Gompertz pattern

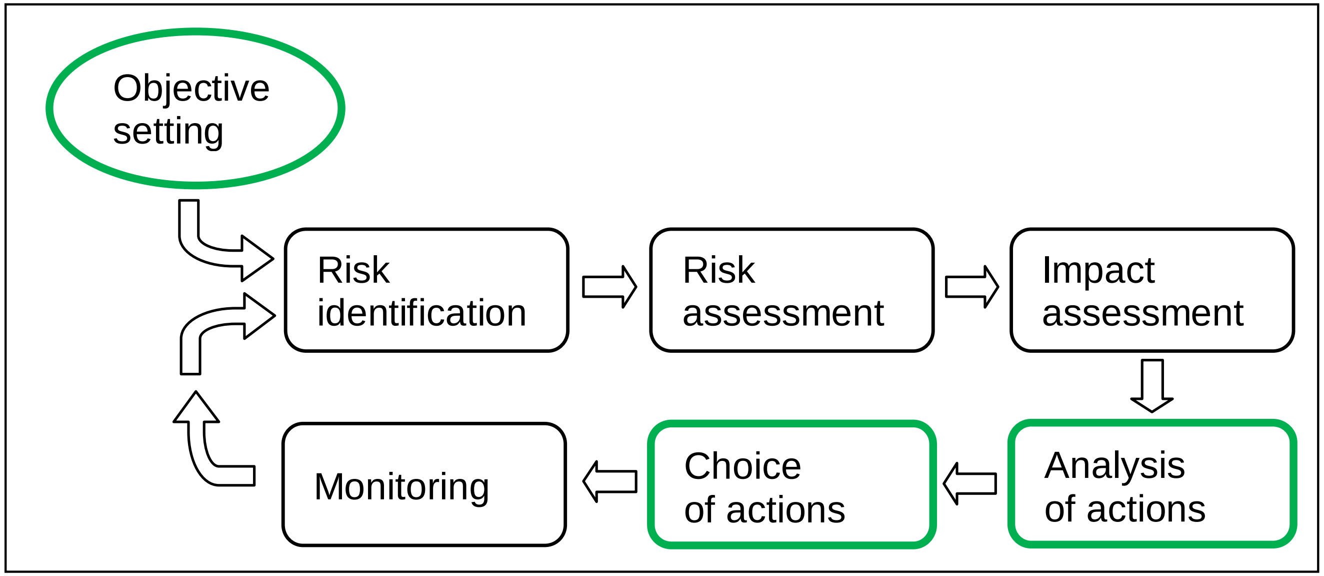
Avraam et al. [2014, 2016]:

- Extend the previous model by allowing for evolution of the parameters $\Rightarrow$ representing mortality trend

6 HETEROGENEITY IN ACTUARIAL EVALUATIONS

In what follows we focus on life annuity portfolios

Back to the RM process:



The RM process (2)

Heterogeneity in actuarial evaluations *(cont'd)*

Objectives	**Actions**
Profit, value creation Market share	Product design, Pricing
Solvency	Capital allocation, Reinsurance, ART

RM Objectives & Actions

PRICING

Does disregarding (unobservable) heterogeneity in a life annuity portfolio leads to wrong pricing ?

A. Olivieri [2006]:

- Refer to a portfolio of life annuities; all annuitants aged $x = 65$ initially; group is closed to new entrants; death is the only cause of decrement; same annual benefit b paid to all the annuitants
- Mortality in the portfolio according to the Gompertz-Gamma model $\Rightarrow$ fixed individual frailty
- Main results: disregarding heterogeneity in the portfolio leads to
  - ▷ underestimation of the actuarial values and hence, in particular, of premiums and policy reserves
  - ▷ underestimation of the (relative) riskiness in the portfolio (expressed by the coefficient of variation) $\Rightarrow$ and underestimation of the adequacy requirements, in terms of risk margin and/or solvency capital

Su and Sherris [2012]:

- Refer to a portfolio of life annuities
- Mortality in the portfolio alternatively given by:
  1. fixed individual frailty, Gamma distributed or Inverse Gaussian distributed
  2. Markov ageing model
- Both models for heterogeneity have implications for annuity markets
- Extent to which life annuity rates vary with age shows the financial significance of heterogeneity implied by the models

CAPITAL ALLOCATION

What is an appropriate capital allocation in face of a heterogeneous life annuity portfolio ?

Sherris and Zhou [2014]:

- *Biometric risk components* in a life annuity portfolio
  - ▷ idiosyncratic longevity risk $\Rightarrow$ diversifiable via risk pooling
  - ▷ aggregate longevity risk $\Rightarrow$ systematic risk, non-diversifiable via risk pooling
  - ▷ heterogeneity w.r.t. mortality $\Rightarrow$ weakens the diversification of idiosyncratic longevity risk
- Heterogeneity alternatively represented by fixed-frailty model and dynamic model
- Main result: increasing pool sizes increases tail risk when a mortality model includes systematic risk $\Rightarrow$ higher capital allocation required
  - ▷ effect not captured by standard models of heterogeneity

ART

Alternative Risk Transfers (swaps, mortality-linked securities, etc.) must be implemented in order to hedge biometric risks non-diversifiable via pooling

Liu and Lin [2012]:

- A subordinated Markov model is adopted for modeling stochastic mortality
  - ▷ the aging process of a life is assumed to follow a finite-state Markov process
  - ▷ stochasticity of mortality is governed by a subordinating gamma process
- The model is applied to the evaluation of mortality-linked securities hedging the longevity risk, i.e. longevity bonds

PRODUCT DESIGN

Can heterogeneity in a population of potential annuitants suggest rating procedures in order to enlarge annuity portfolios ?

$\Rightarrow$ Definition of underwriting procedures $\Rightarrow$ *underwritten annuities*, or *special-rate annuities*, i.e. life annuities with lower premiums for individuals in non-optimal health conditions (see, for example, Pitacco [2017] and references therein)

Meyricke and Sherris [2013]:

- Standard annuities are priced assuming above-average longevity
- Underwritten annuity prices reflect individual risk factors based on underwriting
- Mortality risk still varies within each risk class due to unobservable individual risk factors (frailty)
- The paper quantifies the impact of heterogeneity due to underwriting factors and frailty on annuity values

Olivieri and Pitacco [2016]:

- Portfolio consisting of standard annuities and special-rate annuities
- Larger size $\Rightarrow$ contributes to lower variance in portfolio results (as regards the idiosyncratic risk, i.e. risk of random fluctuations)
- Heterogeneity in the combined portfolio $\Rightarrow$ contributes to raise variance in portfolio results
  - ▷ heterogeneity among sub-portfolios
  - ▷ some degree of residual heterogeneity inside each sub-portfolio, because of residual non-observable risk factors (the underwriting process only provides a proxy)
- What about the "balance"?
- Numerical examples show that appropriate rating classes can improve the portfolio risk profile

7 CONCLUDING REMARKS

We have focused on:

- Awareness of heterogeneity w.r.t. mortality, due to
  - ▷ observable risk factors
  - ▷ unobservable risk factors
- Modeling approaches
- Actuarial implications of heterogeneity, specifically due to unobservable risk factors, in particular:
  - ▷ risk profile of heterogeneous life annuity portfolios

Special attention should be placed on assessing and hedging biometric risks (and tail risk in particular) in a complex framework allowing for all the risk components

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Where links are provided, they were active as of the time this presentation was completed but may have been updated since then

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Many thanks

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