Risk Aggregation with Dependence Uncertainty

Carole Bernard

GEM and VUB

Risk: Modelling, Optimization and Inference with Applications in Finance, Insurance and Superannuation

Sydney
December 7-8, 2017
Motivation on VaR aggregation with dependence uncertainty

**Full** information on marginal distributions:

\[ X_j \sim F_j \]

\[
\]

**Full** Information on dependence:

(known copula)

\[
\Rightarrow
\]

\[ \text{VaR}_q (X_1 + X_2 + \ldots + X_d) \] can be computed!
Motivation on VaR aggregation with dependence uncertainty

**Full** information on marginal distributions:

\[ X_j \sim F_j \]

\[ + \]

**Partial or no** Information on dependence:

(incomplete information on copula)

\[ \Rightarrow \]

\[ \text{VaR}_q (X_1 + X_2 + ... + X_d) \text{ cannot} \] be computed!

Only a range of possible values for \( \text{VaR}_q (X_1 + X_2 + ... + X_d) \).
Acknowledgement of Collaboration

with M. Denuit (UCL), X. Jiang (UW), L. Rüschendorf (Freiburg), S. Vanduffel (VUB), J. Yao (VUB), R. Wang (UW):

Model Risk

1. Goal: Assess the risk of a portfolio sum $S = \sum_{i=1}^{d} X_i$.

2. Choose a risk measure $\rho(\cdot)$: variance, Value-at-Risk...

3. “Fit” a multivariate distribution for $(X_1, X_2, ..., X_d)$ and compute $\rho(S)$

4. How about model risk? How wrong can we be?
Model Risk

1. Goal: Assess the risk of a portfolio sum \( S = \sum_{i=1}^{d} X_i \).
2. Choose a risk measure \( \rho(\cdot) \): variance, Value-at-Risk...
3. “Fit” a multivariate distribution for \((X_1, X_2, \ldots, X_d)\) and compute \( \rho(S) \)
4. How about model risk? How wrong can we be?

Assume \( \rho(S) = \text{var}(S) \),

\[
\rho^+_\mathcal{F} := \sup \left\{ \text{var} \left( \sum_{i=1}^{d} X_i \right) \right\}, \quad \rho^-_{\mathcal{F}} := \inf \left\{ \text{var} \left( \sum_{i=1}^{d} X_i \right) \right\}
\]

where the bounds are taken over all other (joint distributions of) random vectors \((X_1, X_2, \ldots, X_d)\) that “agree” with the available information \(\mathcal{F}\)
Aggregation with dependence uncertainty: Example - Credit Risk

- Marginals known
- Dependence fully unknown

Consider a portfolio of 10,000 loans all having a default probability \( p = 0.049 \).

<table>
<thead>
<tr>
<th>( q )</th>
<th>Min ( \text{VaR}_q )</th>
<th>Max ( \text{VaR}_q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q = 0.95 )</td>
<td>0%</td>
<td>98%</td>
</tr>
<tr>
<td>( q = 0.995 )</td>
<td>4.4%</td>
<td>100%</td>
</tr>
</tbody>
</table>

Portfolio models are subject to significant model uncertainty (defaults are rare and correlated events).
Aggregation with dependence uncertainty: Example - Credit Risk

- **Marginals known**

- **Dependence fully unknown**

Consider a portfolio of 10,000 loans all having a default probability $p = 0.049$. The default correlation is $\rho = 0.0157$ (for KMV).

| q = 0.95  | 10.1% | 0%  | 98% |
| q = 0.995 | 15.1% | 4.4% | 100% |

Portfolio models are subject to significant model uncertainty (defaults are rare and correlated events). Using dependence information is crucial to try to get more “reasonable” bounds.
Objectives and Findings

- Model uncertainty on the risk assessment of an aggregate portfolio: the sum of $d$ dependent risks.
  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of a portfolio?
Objectives and Findings

- Model uncertainty on the risk assessment of an aggregate portfolio: the sum of $d$ dependent risks.
  - Given all information available in the market, what can we say about the maximum and minimum possible values of a given risk measure of a portfolio?

- Findings / Implications:
  - Current VaR based regulation is subject to high model risk, even if one knows the multivariate distribution “almost completely”.
Outline of the Talk

Part 1: The Rearrangement Algorithm

- Minimizing variance of a sum with full dependence uncertainty
- Variance bounds
- With partial dependence information on a subset

Part 2: Application to Uncertainty on Value-at-Risk

- With 2 risks and full dependence uncertainty
- With $d$ risks and full dependence uncertainty
- With partial dependence information on a subset

Part 3: Other extensions: alternative information on dependence
Part I

The Rearrangement Algorithm

Portfolio with minimum variance
Risk Aggregation and full dependence uncertainty

- Marginals known:
- Dependence fully unknown
- In two dimensions $d = 2$, assessing model risk on variance is linked to the Fréchet-Hoeffding bounds
  $$\text{var}(F_1^{-1}(U) + F_2^{-1}(1-U)) \leq \text{var}(X_1 + X_2) \leq \text{var}(F_1^{-1}(U) + F_2^{-1}(U))$$
- Maximum variance is obtained for the comonotonic scenario:
  $$\text{var}(X_1 + X_2 + \ldots + X_d) \leq \text{var}(F_1^{-1}(U) + F_2^{-1}(U) + \ldots + F_d^{-1}(U))$$
- Minimum variance: A challenging problem in $d \geq 3$ dimensions
  - Wang and Wang (2011, JMVA): concept of complete mixability
  - Puccetti and Rüschendorf (2012): algorithm (RA) useful to approximate the minimum variance.
Rearrangement Algorithm

$N = 4$ observations of $d = 3$ variables: $X_1, X_2, X_3$

$$M = \begin{bmatrix}
1 & 1 & 2 \\
0 & 6 & 3 \\
4 & 0 & 0 \\
6 & 3 & 4
\end{bmatrix}$$

Each column: **marginal** distribution.
Interaction among columns: **dependence** among the risks.
Same marginals, different dependence ⇒ Effect on the sum!

\[ M = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 6 & 3 \\ 4 & 0 & 0 \\ 6 & 3 & 4 \end{bmatrix} \]

\[ S_N = \begin{bmatrix} 4 \\ 9 \\ 4 \\ 13 \end{bmatrix} \]

\[ X_1 + X_2 + X_3 \]

\[ S_N = \begin{bmatrix} 16 \\ 10 \\ 3 \\ 0 \end{bmatrix} \]

Aggregate Risk with Maximum Variance
comonotonic scenario \( S^c \)
Rearrangement Algorithm: Sum with Minimum Variance

Minimum variance with \( d = 2 \) risks \( X_1 \) and \( X_2 \)

Antimonotonicity: \( \text{var}(X_1^a + X_2) \leq \text{var}(X_1 + X_2) \).

How about in \( d \) dimensions?
Rearrangement Algorithm: Sum with Minimum Variance

minimum variance with $d = 2$ risks $X_1$ and $X_2$

Antimonotonicity: $\text{var}(X_1 + X_2) \leq \text{var}(X_1 + X_2)$.

How about in $d$ dimensions?
Use of the rearrangement algorithm on the original matrix $M$.

Aggregate Risk with Minimum Variance

- Columns of $M$ are rearranged such that they become anti-monotonic with the sum of all other columns:

  $\forall k \in \{1, 2, \ldots, d\}, X_k^a$ antimonotonic with $\sum_{j \neq k} X_j$.

- After each step, $\text{var} \left( X_k^a + \sum_{j \neq k} X_j \right) \leq \text{var} \left( X_k + \sum_{j \neq k} X_j \right)$
  where $X_k^a$ is antimonotonic with $\sum_{j \neq k} X_j$. 
A New Approach to Assessing Model Risk in High Dimensions

Carole Bernard∗ and Steven Vanduffel†‡

July 14, 2014

\[
M = \begin{bmatrix}
1 & 1 & 2 \\
0 & 6 & 3 \\
4 & 0 & 0 \\
6 & 3 & 4
\end{bmatrix}
\]

\[
S_N = \begin{bmatrix}
4 & 9 & 4 \\
13 & 16 & 9 \\
3 & 0 & 0
\end{bmatrix}
\]

Aggregate risk with minimum variance

Step 1: First column

\[X_2 + X_3\]

\[
\begin{bmatrix}
6 & 6 & 4 \\
4 & 3 & 2 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

becomes

\[
\begin{bmatrix}
1 & 3 & 2 \\
4 & 1 & 1 \\
6 & 0 & 0
\end{bmatrix}
\]
Aggregate risk with minimum variance

\( X_2 + X_3 \)

\[
\begin{bmatrix}
6 & 6 & 4 \\
4 & 3 & 2 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\xrightarrow{\downarrow}
\begin{bmatrix}
10 \\
5 \\
2 \\
0
\end{bmatrix}
\]

becomes

\[
\begin{bmatrix}
0 & 6 & 4 \\
1 & 3 & 2 \\
4 & 1 & 1 \\
6 & 0 & 0
\end{bmatrix}
\]

\( X_1 + X_3 \)

\[
\begin{bmatrix}
0 & 6 & 4 \\
1 & 3 & 2 \\
4 & 1 & 1 \\
6 & 0 & 0
\end{bmatrix}
\xrightarrow{\downarrow}
\begin{bmatrix}
4 \\
3 \\
5 \\
6
\end{bmatrix}
\]

becomes

\[
\begin{bmatrix}
0 & 3 & 4 \\
1 & 6 & 2 \\
4 & 1 & 1 \\
6 & 0 & 0
\end{bmatrix}
\]

\( X_1 + X_2 \)

\[
\begin{bmatrix}
0 & 3 & 4 \\
1 & 6 & 2 \\
4 & 1 & 1 \\
6 & 0 & 0
\end{bmatrix}
\xrightarrow{\downarrow}
\begin{bmatrix}
3 \\
7 \\
5 \\
6
\end{bmatrix}
\]

becomes

\[
\begin{bmatrix}
0 & 3 & 4 \\
1 & 6 & 0 \\
4 & 1 & 2 \\
6 & 0 & 1
\end{bmatrix}
\]
## Aggregate risk with minimum variance

Each column is antimonotonic with the sum of the others:

<table>
<thead>
<tr>
<th>Column</th>
<th>$X_2 + X_3$</th>
<th>$X_1 + X_3$</th>
<th>$X_1 + X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_2$</td>
<td>0 3 4</td>
<td>0 3 4</td>
<td>0 3 4</td>
</tr>
<tr>
<td>$X_3$</td>
<td>1 6 0</td>
<td>1 6 0</td>
<td>1 6 0</td>
</tr>
<tr>
<td>$X_1$</td>
<td>4 1 2</td>
<td>4 1 2</td>
<td>4 1 2</td>
</tr>
<tr>
<td>$X_4$</td>
<td>6 0 1</td>
<td>6 0 1</td>
<td>6 0 1</td>
</tr>
</tbody>
</table>

The minimum variance of the sum is equal to 0! Ideal case of a constant sum (*complete mixability*, see Wang and Wang (2011)).
Bounds on variance

Analytical Bounds on Standard Deviation

Consider $d$ risks $X_i$ with standard deviation $\sigma_i$

$$0 \leq std(X_1 + X_2 + \ldots + X_d) \leq \sigma_1 + \sigma_2 + \ldots + \sigma_d.$$

Example with 20 normal $N(0,1)$

$$0 \leq std(X_1 + X_2 + \ldots + X_{20}) \leq 20,$$

in this case, both bounds are sharp and too wide for practical use!

Our idea: Incorporate information on dependence.
Illustration with 2 risks with marginals $N(0,1)$
Assumption: Independence on $\mathcal{F} = \bigcap_{k=1}^{2} \{ q_{\beta} \leq X_k \leq q_{1-\beta} \}$.
Our assumptions on the cdf of \((X_1, X_2, \ldots, X_d)\)

\[ \mathcal{F} \subset \mathbb{R}^d \] ("trusted" or "fixed" area)
\[ \mathcal{U} = \mathbb{R}^d \setminus \mathcal{F} \] ("untrusted").

We assume that we know:

(i) the marginal distribution \(F_i\) of \(X_i\) on \(\mathbb{R}\) for \(i = 1, 2, \ldots, d\),
(ii) the distribution of \((X_1, X_2, \ldots, X_d)\) \(|\{(X_1, X_2, \ldots, X_d) \in \mathcal{F}\}\).
(iii) \(P((X_1, X_2, \ldots, X_d) \in \mathcal{F})\).

- When only marginals are known: \(\mathcal{U} = \mathbb{R}^d\) and \(\mathcal{F} = \emptyset\).
- **Our Goal:** Find bounds on \(\rho(S) := \rho(X_1 + \ldots + X_d)\) when \((X_1, \ldots, X_d)\) satisfy (i), (ii) and (iii).
Example:

$N = 8$ observations, $d = 3$ dimensions and 3 observations trusted ($p_f = 3/8$).

$$S_N = \begin{bmatrix}
3 & 4 & 1 \\
1 & 1 & 1 \\
0 & 3 & 2 \\
0 & 2 & 1 \\
2 & 4 & 2 \\
3 & 0 & 1 \\
1 & 1 & 2 \\
4 & 2 & 3 \\
\end{bmatrix} = \begin{bmatrix}
8 \\
3 \\
5 \\
3 \\
8 \\
4 \\
4 \\
9 \\
\end{bmatrix}$$
Example: \( N = 8, \ d = 3 \) with 3 observations trusted

**Maximum variance:**

\[
M = \begin{bmatrix}
3 & 4 & 1 \\
2 & 4 & 2 \\
0 & 2 & 1 \\
4 & 3 & 3 \\
3 & 2 & 2 \\
1 & 1 & 2 \\
1 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad S_N^f = \begin{bmatrix}
8 \\
8 \\
3 \\
\end{bmatrix}, \quad S_N^u = \begin{bmatrix}
10 \\
7 \\
4 \\
3 \\
1 \\
\end{bmatrix}
\]

**Minimum variance:**

\[
M = \begin{bmatrix}
3 & 4 & 1 \\
2 & 4 & 2 \\
0 & 2 & 1 \\
1 & 1 & 3 \\
0 & 3 & 2 \\
1 & 2 & 2 \\
3 & 1 & 1 \\
4 & 0 & 1 \\
\end{bmatrix}, \quad S_N^f = \begin{bmatrix}
8 \\
8 \\
3 \\
\end{bmatrix}, \quad S_N^u = \begin{bmatrix}
5 \\
5 \\
5 \\
\end{bmatrix}
\]
Example $d = 20$ risks $N(0,1)$

$$(X_1, ..., X_{20}) \text{ independent } N(0,1) \text{ on}$$

$$\mathcal{F} := [q_\beta, q_{1-\beta}]^d \subset \mathbb{R}^d \quad p_f = P \left((X_1, ..., X_{20}) \in \mathcal{F}\right)$$

(for some $\beta \leq 50\%$) where $q_\gamma$: $\gamma$-quantile of $N(0,1)$.

$\beta = 0\%$: no uncertainty (20 independent $N(0,1)$).

$\beta = 50\%$: full uncertainty.

<table>
<thead>
<tr>
<th>$\mathcal{F} = [q_\beta, q_{1-\beta}]^d$</th>
<th>$U = \emptyset$</th>
<th>$\beta = 0%$</th>
<th>$\beta = 50%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho = 0$</td>
<td></td>
<td>4.47</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0, 20)</td>
</tr>
</tbody>
</table>
Example $d = 20$ risks $N(0,1)$

$\Rightarrow$ $(X_1, \ldots, X_{20})$ independent $N(0,1)$ on

$$\mathcal{F} := [q_\beta, q_{1-\beta}]^d \subset \mathbb{R}^d \quad p_f = P((X_1, \ldots, X_{20}) \in \mathcal{F})$$

(for some $\beta \leq 50\%$) where $q_\gamma$: $\gamma$-quantile of $N(0,1)$

$\Rightarrow$ $\beta = 0\%$: no uncertainty (20 independent $N(0,1)$)

$\Rightarrow$ $\beta = 50\%$: full uncertainty

<table>
<thead>
<tr>
<th>$\mathcal{F} = [q_\beta, q_{1-\beta}]^d$</th>
<th>$\mathcal{U} = \emptyset$</th>
<th>$p_f \approx 98%$</th>
<th>$p_f \approx 82%$</th>
<th>$\mathcal{U} = \mathbb{R}^d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = 0%$</td>
<td>$\beta = 0%$</td>
<td>$\beta = 0.05%$</td>
<td>$\beta = 0.5%$</td>
<td>$\beta = 50%$</td>
</tr>
<tr>
<td>$\rho = 0$</td>
<td>$4.47$</td>
<td>$(4.4, 5.65)$</td>
<td>$(3.89, 10.6)$</td>
<td>$(0, 20)$</td>
</tr>
</tbody>
</table>

Model risk on the volatility of a portfolio is reduced a lot by incorporating information on dependence!
Information on the joint distribution

- Can come from a fitted model
- Can come from experts’ opinions
- Dependence “known” on specific scenarios
Illustration with marginals $N(0,1)$
Illustration with marginals $N(0,1)$

\[ \mathcal{F}_1 = \bigcap_{k=1}^{2} \left\{ q_\beta \leq X_k \leq q_{1-\beta} \right\} \]
Illustration with marginals \( N(0,1) \)

\[
\mathcal{F}_1 = \bigcap_{k=1}^{2} \{ q_\beta \leq X_k \leq q_{1-\beta} \}
\]

\[
\mathcal{F} = \bigcup_{k=1}^{2} \{ X_k > q_p \} \bigcup \mathcal{F}_1
\]
Illustration with marginals N(0,1)

\[ \mathcal{F}_1 = \text{contour of MVN at } \beta \]

\[ \mathcal{F} = \bigcup_{k=1}^{2} \{X_k > q_p\} \bigcup \mathcal{F}_1 \]
Conclusions of Part I

- **Part 1:** Assess model risk for variance of a portfolio of risks with given marginals but partially known dependence.

- Same method applies to TVaR (expected Shortfall) or any risk measure that satisfies convex order (but not for Value-at-Risk).

- **Challenges:**
  - Choosing the trusted area $\mathcal{F}$
  - $N$ too small: possible to improve the efficiency of the algorithm by re-discretizing using the fitted marginal $\hat{f}_i$.
  - Possible to amplify the tails of the marginals
Part II

Another application of the Rearrangement Algorithm

VaR aggregation with dependence uncertainty

- Maximum Value-at-Risk is not caused by the comonotonic scenario.
- Maximum Value-at-Risk is achieved when the variance is *minimum* in the tail. The RA is then used in the tails only.
- Bounds on Value-at-Risk at high confidence level stay wide even when the trusted area covers 98% of the space!
Risk Aggregation and full dependence uncertainty

Literature review

- Marginals known
- Dependence fully unknown

- Explicit sharp (attainable) bounds
  - $n = 2$ (Makarov (1981), Rüschendorf (1982))

- A challenging problem in $n \geq 3$ dimensions

- Approximate sharp bounds
  - Puccetti and Rüschendorf (2012): algorithm (RA) useful to approximate the minimum variance.
  - Embrechts, Puccetti, Rüschendorf (2013): algorithm (RA) to find bounds on VaR
Our Contributions

► Issues

- bounds on VaR are generally very wide
- ignore all information on dependence.
Our Contributions

▶ Issues
  • bounds on VaR are generally very wide
  • ignore all information on dependence.

▶ Our contributions:
  • Incorporating in a natural way dependence information.
  • Getting simple upper and lower bounds for VaR (not sharp in general)
  • Extend the RA to deal with additional dependence information
VaR Bounds with full dependence uncertainty

(Unconstrained VaR bounds)
“Riskiest” Dependence: maximum $\text{VaR}_q$ in 2 dims?

If $X_1$ and $X_2$ are $\text{U}(0,1)$ comonotonic, then

$$\text{VaR}_q(S^c) = \text{VaR}_q(X_1) + \text{VaR}_q(X_2) = 2q.$$
“Riskiest” Dependence: maximum $\text{VaR}_q$ in 2 dims?

If $X_1$ and $X_2$ are $U(0,1)$ comonotonic, then

$$\text{VaR}_q(S^c) = \text{VaR}_q(X_1) + \text{VaR}_q(X_2) = 2q.$$ 

Note that

$$TV\text{VaR}_q(S^c) = \frac{\int_q^1 2pdq}{1-q} = 1 + q.$$
“Riskiest” Dependence: maximum \( \text{VaR}_q \) in 2 dims

If \( X_1 \) and \( X_2 \) are \( \text{U}(0,1) \) and antimonotonic in the tail, then \( \text{VaR}_q(S^*) = 1 + q \) (which is maximum possible).

\[
\text{VaR}_q(S^*) = 1 + q > \text{VaR}_q(S^c) = 2q
\]

\( \Rightarrow \) to maximize \( \text{VaR}_q \), the idea is to change the comonotonic dependence such that the sum is constant in the tail
VaR at level $q$ of the comonotonic sum w.r.t. $q$
VaR at level $q$ of the comonotonic sum w.r.t. $q$

where $\text{TVaR}$ (Expected shortfall): $\text{TVaR}_q(X) = \frac{1}{1 - q} \int_q^1 \text{VaR}_u(X) du$
Riskiest Dependence Structure VaR at level $q$

$S^* \Rightarrow \text{VaR}_q(S^*) = \text{TVaR}_q(S^c)$?
Analytic expressions (not sharp)

Analytical Unconstrained Bounds with $X_j \sim F_j$

\[ A = LTVaR_q(S^c) \leq \text{Var}_q [X_1 + X_2 + \ldots + X_n] \leq B = TVaR_q(S^c) \]
VaR Bounds with full dependence uncertainty

Approximate sharp bounds:

- Puccetti and Rüschendorf (2012): algorithm (RA) useful to approximate the minimum variance.
- Embrechts, Puccetti, Rüschendorf (2013): algorithm (RA) to find bounds on VaR
Illustration for the maximum $\text{VaR}_q$ (1/3)
Illustration for the maximum VaR$_q$ (2/3)

Rearrange within columns..to make the sums as constant as possible...
B=(11+15+25+29)/4=20
Illustration for the maximum VaR$_q$ (3/3)

\[
\begin{pmatrix}
8 & 8 & 4 \\
10 & 7 & 3 \\
12 & 1 & 7 \\
11 & 0 & 9 \\
\end{pmatrix}
\]

\[\text{Sum} = 20\]

\[\text{Sum} = 20\]

\[\text{Sum} = 20\]

\[\text{Sum} = 20\]

\[= B!\]
Adding information

Information on a subset

VaR bounds when the joint distribution of $(X_1, X_2, ..., X_n)$ is known on a subset of the sample space.
Numerical Results for VaR, 20 risks $N(0,1)$

When marginal distributions are given,

- What is the maximum Value-at-Risk?
- What is the minimum Value-at-Risk?
- A portfolio of 20 risks normally distributed $N(0,1)$. Bounds on $\text{VaR}_q$ (by the rearrangement algorithm applied on each tail)

<table>
<thead>
<tr>
<th>$q$</th>
<th>Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>95%</td>
<td>(-2.17, 41.3)</td>
</tr>
<tr>
<td>99.95%</td>
<td>(-0.035, 71.1)</td>
</tr>
</tbody>
</table>


- Very wide bounds
- All dependence information ignored

**Idea:** add information on dependence from a fitted model or from experts’ opinions
Our assumptions on the cdf of \((X_1, X_2, \ldots, X_n)\)

\[ \mathcal{F} \subset \mathbb{R}^n \] ("trusted" or "fixed" area)

\[ \mathcal{U} = \mathbb{R}^n \setminus \mathcal{F} \] ("untrusted").

We assume that we know:

(i) the marginal distribution \(F_i\) of \(X_i\) on \(\mathbb{R}\) for \(i = 1, 2, \ldots, n\),

(ii) the distribution of \((X_1, X_2, \ldots, X_n) | \{(X_1, X_2, \ldots, X_n) \in \mathcal{F}\}\).

(iii) \(P((X_1, X_2, \ldots, X_n) \in \mathcal{F})\)

\[ \textbf{Goal:} \text{ Find bounds on } \text{VaR}_q(S) := \text{VaR}_q(X_1 + \ldots + X_n) \]

\text{when } (X_1, \ldots, X_n) \text{ satisfy (i), (ii) and (iii).}
Numerical Results, 20 correlated $N(0, 1)$ on $\mathcal{F} = [q_\beta, q_{1-\beta}]^n$

<table>
<thead>
<tr>
<th>$\mathcal{F}$</th>
<th>$\mathcal{U} = \emptyset$</th>
<th>$\mathcal{U} = \mathbb{R}^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q=95%$</td>
<td>$\beta = 0%$</td>
<td>12.5</td>
</tr>
<tr>
<td>$q=99.5%$</td>
<td>19.6</td>
<td>( -0.29 , 57.8 )</td>
</tr>
<tr>
<td>$q=99.95%$</td>
<td>25.1</td>
<td>( -0.035 , 71.1 )</td>
</tr>
</tbody>
</table>

- $\mathcal{U} = \emptyset$: 20 correlated standard normal variables ($\rho = 0.1$).

$$\text{VaR}_{95\%} = 12.5 \quad \text{VaR}_{99.5\%} = 19.6 \quad \text{VaR}_{99.95\%} = 25.1$$
Numerical Results, 20 correlated $N(0, 1)$ on $\mathcal{F} = [q_\beta, q_{1-\beta}]^n$

<table>
<thead>
<tr>
<th>$\mathcal{U}$</th>
<th>$p_f \approx 98%$</th>
<th>$p_f \approx 82%$</th>
<th>$p_f \approx 98%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$, $\beta = 0%$</td>
<td>12.5</td>
<td>(12.2, 13.3)</td>
<td>(10.7, 27.7)</td>
</tr>
<tr>
<td>$\emptyset$, $\beta = 0.05%$</td>
<td>19.6</td>
<td>(19.1, 31.4)</td>
<td>(16.9, 57.8)</td>
</tr>
<tr>
<td>$\emptyset$, $\beta = 0.5%$</td>
<td>25.1</td>
<td>(24.2, 71.1)</td>
<td>(21.5, 71.1)</td>
</tr>
<tr>
<td>$\emptyset$, $\beta = 50%$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- $\mathcal{U} = \emptyset$: 20 correlated standard normal variables ($\rho = 0.1$).

\[
\text{VaR}_{95\%} = 12.5 \quad \text{VaR}_{99.5\%} = 19.6 \quad \text{VaR}_{99.95\%} = 25.1
\]

- The risk for an underestimation of VaR is increasing in the probability level used to assess the VaR.
- For VaR at high probability levels ($q = 99.95\%$), despite all the added information on dependence, the bounds are still wide!
Conclusions

We have shown that

- Maximum Value-at-Risk is not caused by the comonotonic scenario.
- Maximum Value-at-Risk is achieved when the variance is minimum in the tail. The RA is then used in the tails only.
- Bounds on Value-at-Risk at high confidence level stay wide even if the multivariate dependence is known in 98% of the space!

▷ Assess model risk with partial information and given marginals
▷ Design algorithms for bounds on variance, TVaR and VaR and many more risk measures.
▷ A regulation challenge...
The Basel Committee (2013) insists that a desired objective of a Solvency framework concerns comparability:

“Two banks with portfolios having identical risk profiles apply the framework’s rules and arrive at the same amount of risk-weighted assets, and two banks with different risk profiles should produce risk numbers that are different proportionally to the differences in risk”
Part III

VaR Bounds with partial dependence uncertainty

VaR Bounds with other types of Dependence Information...
Adding dependence information

Finding minimum and maximum possible values for VaR of the credit portfolio loss, \( S = \sum_{i=1}^{n} X_i \), given that

- known marginal distributions of the risks \( X_i \).
- some dependence information.

**Example 1:** Variance constraint - with Rüschendorf and Vanduffel

\[
M := \sup \text{VaR}_q [X_1 + X_2 + ... + X_n], \\
\text{subject to } X_j \sim F_j, \text{var}(X_1 + X_2 + ... + X_n) \leq s^2
\]

**Example 2:** Moments constraint - with Denuit, Rüschendorf, Vanduffel, Yao

\[
M := \sup \text{VaR}_q [X_1 + X_2 + ... + X_n], \\
\text{subject to } X_j \sim F_j, E(X_1 + X_2 + ... + X_n)^k = c_k
\]
Adding dependence information

Example 3: with Rüschendorf, Vanduffel and Wang

\[ M := \sup \text{VaR}_q [X_1 + X_2 + \ldots + X_n], \]
subject to \((X_j, Z) \sim H_j,\]

where \(Z\) is a factor.
Examples

Example 1: variance constraint

\[ M := \sup \text{VaR}_q [X_1 + X_2 + \ldots + X_n], \]
subject to \( X_j \sim F_j, \text{var}(X_1 + X_2 + \ldots + X_n) \leq s^2 \)

Example 2: Moments constraint

\[ M := \sup \text{VaR}_q [X_1 + X_2 + \ldots + X_n], \]
subject to \( X_j \sim F_j, \text{E}(X_1 + X_2 + \ldots + X_n)^k \leq c_k \)
for all \( k \) in 2,\ldots,K
Without moment constraints, VaR bounds are attained if there exists a dependence among risks $X_i$ such that

$$S = \begin{cases} A \text{ probability } q \\ B \text{ probability } 1 - q \end{cases} \quad \text{a.s.}$$

If the “distance” between $A$ and $B$ is too wide then improved bounds are obtained with

$$S^* = \begin{cases} a \text{ with probability } q \\ b \text{ with probability } 1 - q \end{cases}$$

such that

$$\begin{cases} a^k q + b^k (1 - q) \leq c_k \\ a q + b (1 - q) = E[S] \end{cases}$$

in which $a$ and $b$ are “as distant as possible while satisfying all constraints” (for all $k$)
Analytical result for variance constraint

$A$ and $B$: unconstrained bounds on Value-at-Risk, $\mu = E[S]$.

Constrained Bounds with $X_j \sim F_j$ and variance $\leq s^2$

$$a = \max \left( A, \mu - s\sqrt{\frac{1-q}{q}} \right) \leq \text{VaR}_q [X_1 + X_2 + \ldots + X_n]$$

$$\leq b = \min \left( B, \mu + s\sqrt{\frac{q}{1-q}} \right)$$

- If the variance $s^2$ is not “too large” (i.e. when $s^2 \leq q(A - \mu)^2 + (1-q)(B - \mu)^2$), then $b < B$.
- The “target” distribution for the sum: a two-point cdf that takes two values $a$ and $b$. We can write

$$X_1 + X_2 + \ldots + X_n - S = 0$$

and apply the standard RA.
**Extended RA**

<table>
<thead>
<tr>
<th></th>
<th>8</th>
<th>8</th>
<th>4</th>
<th>-b</th>
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<tr>
<td>1-q</td>
<td>10</td>
<td>7</td>
<td>3</td>
<td>-b</td>
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<tr>
<td>1-q</td>
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<tr>
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<td></td>
<td></td>
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</tbody>
</table>

Rearrange now within all columns such that all sums become close to zero.
Corporate portfolio

- a corporate portfolio of a major European Bank.
- 4495 loans mainly to medium sized and large corporate clients
- total exposure (EAD) is 18642.7 (million Euros), and the top 10% of the portfolio (in terms of EAD) accounts for 70.1% of it.
- portfolio exhibits some heterogeneity.

<table>
<thead>
<tr>
<th>Summary statistics of a corporate portfolio</th>
</tr>
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<tbody>
<tr>
<td></td>
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<tr>
<td>Default probability</td>
</tr>
<tr>
<td>---------------------</td>
</tr>
<tr>
<td>0.0001 0.15 0.0119</td>
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<tr>
<td>EAD 0 750.2 116.7</td>
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<tr>
<td>LGD 0 0.90 0.41</td>
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</table>
## Comparison of Industry Models

<table>
<thead>
<tr>
<th>$q = \rho = 0.10$</th>
<th>Comon.</th>
<th>KMV</th>
<th>Credit Risk$^+$</th>
<th>Beta</th>
</tr>
</thead>
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<tr>
<td>95% 99% 99.5%</td>
<td>393.5  2374.1  5088.5</td>
<td>340.6  539.4  631.5</td>
<td>346.2  513.4  582.9</td>
<td>347.4  520.2  593.5</td>
</tr>
</tbody>
</table>

VaRs of the corporate portfolio under different industry models.
VaR bounds

Model risk assessment of the VaR of the corporate portfolio

(we use $\rho = 0.1$ to construct moments constraints)

<table>
<thead>
<tr>
<th>$q =$</th>
<th>KMV</th>
<th>Comon.</th>
<th>Unconstrained</th>
<th>$K = 2$</th>
<th>$K = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>95%</td>
<td>340.6</td>
<td>393.3</td>
<td>(34.0 ; 2083.3)</td>
<td>(97.3 ; 614.8)</td>
<td>(100.9 ; 562.8)</td>
</tr>
<tr>
<td>99%</td>
<td>539.4</td>
<td>2374.1</td>
<td>(56.5 ; 6973.1)</td>
<td>(111.8 ; 1245)</td>
<td>(115.0 ; 941.2)</td>
</tr>
<tr>
<td>99.5%</td>
<td>631.5</td>
<td>5088.5</td>
<td>(89.4 ; 10120)</td>
<td>(114.9 ; 1709)</td>
<td>(117.6 ; 1177.8)</td>
</tr>
</tbody>
</table>

- Obs 1: Comparison with analytical bounds
- Obs 2: Significant bounds reduction with moments information
- Obs 3: Significant model risk
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