Portfolio Optimization in Stochastic Environment

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Joint work with Ronnie Sircar and Thaleia Zariphopoulou

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- Very general analysis for semimartingale models: Kramkov & Schachermayer (1999).

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- Viewing the incomplete market problem as a perturbation around a complete markets problem.

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\begin{aligned}
dS_t &= \mu(Y_t)S_t\,dt + \sigma(Y_t)S_t\,dW_t^{(1)} \\
dY_t &= \frac{1}{\varepsilon}b(Y_t)\,dt + \frac{1}{\sqrt{\varepsilon}}a(Y_t)\left(\rho\,dW_t^{(1)} + \sqrt{1-\rho^2}\,dW_t^{(2)}\right),
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- That is: fast mean-reverting stochastic factor as illustrated next:

Fast Mean-Reverting Stochastic Volatility: $\kappa = 1/\varepsilon$

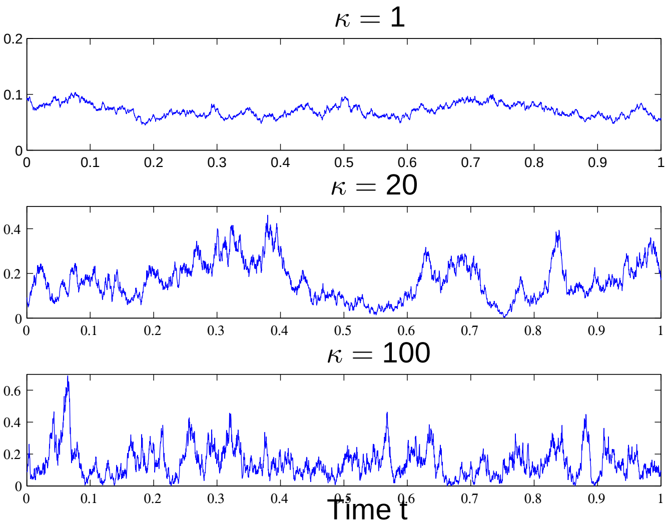

Figure: Simulated CIR Stochastic Volatility (Heston)

Summary of Option Pricing Asymptotics

- Assume a market-selected risk-neutral pricing measure $P^\star$ under which S is a martingale, (S, Y) are jointly Markov and Y is Markov by itself:

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dS_t &= \sigma(Y_t) S_t \, dW_t^{\star(1)} \\
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&\quad + \frac{1}{\sqrt{\varepsilon}} a(Y_t) \left(\rho \, dW_t^{\star(1)} + \sqrt{1-\rho^2} \, dW_t^{\star(2)} \right),
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- European option price:

$$
C^\varepsilon(t, S, y) = E^\star\{ h(S_T) \mid S_t = S, Y_t = y \}.
$$

Reference on Option Pricing Asymptotics

Multiscale Stochastic Volatility for Equity, Interest-Rate and Credit Derivatives
J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Sølna

Cambridge University Press 2011

Option Pricing Asymptotics

- Then for small ε, $C^{\varepsilon}(t,S,y) \approx C_{BS}(t,S) + C_1(t,S)$, where C_{BS} is the Black-Scholes price with the stochastic volatility replaced by the average $\bar{\sigma}$: $\bar{\sigma}^2 = \langle \sigma(\cdot)^2 \rangle$.
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- That is, $\mathcal{L}_{BS} C_{BS} = 0$, with $C_{BS}(T,S) = h(S)$, where

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2 \mathcal{D}_2, \qquad \mathcal{D}_k = S^k \frac{\partial^k}{\partial S^k}.$$

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- The correction C_1 solves

$$\mathcal{L}_{BS} C_1 = -\left(V_2^{\varepsilon}\mathcal{D}_2 + V_3^{\varepsilon}\mathcal{D}_1\mathcal{D}_2\right) C_{BS}, \qquad C_1(T,S) = 0,$$

where $V_2^{\varepsilon}, V_3^{\varepsilon} \propto \sqrt{\varepsilon}$. The V_2^{ε} contains the volatility risk premium; the V_3^{ε} contains the correlation (or skew) ρ.

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  where $V_2^\varepsilon, V_3^\varepsilon \propto \sqrt{\varepsilon}$. The V_2^ε contains the volatility risk premium; the V_3^ε contains the correlation (or skew) ρ.
- The solution is given by

$$C_1 = (T-t)\left(V_2^\varepsilon \mathcal{D}_2 + V_3^\varepsilon \mathcal{D}_1 \mathcal{D}_2\right) C_{BS}$$

  because $\mathcal{L}_{BS} \mathcal{D}_2 = \mathcal{D}_2 \mathcal{L}_{BS}$.

Merton Problem: Wealth Process, Value Function (Work with Ronnie Sircar & Thaleia Zariphopoulou)

- Let X denote the wealth process:

$$dX_t = \pi_t \frac{dS_t}{S_t} + r(X_t - \pi_t)\,dt,$$

  that is, taking $r = 0$ for simplicity,

$$dX_t = \pi_t \mu(Y_t)\,dt + \pi_t \sigma(Y_t)\,dW_t^{(1)}.$$

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- Given a utility function $U(x)$ on $\mathbb{R}_+$ with

$$U'(0^+) = \infty, \quad U'(\infty) = 0 \quad \text{and} \quad \mathrm{AE}[U] := \lim_{x\to\infty} x\frac{U'(x)}{U(x)} < 1,$$

define the value function

$$V(t,x,y) = \sup_{\pi} \mathbb{E}\left\{U(X_T) \mid X_t = x, Y_t = y\right\}.$$

Hamilton-Jacobi-Bellman PDE (quadratic in π)
Injecting π^*, the associated HJB equation is

$$V_t + \frac{1}{\varepsilon}\mathcal{L}_0 V - \frac{\left(\lambda(y) V_x + \frac{\rho a(y)}{\sqrt{\varepsilon}} V_{xy}\right)^2}{2V_{xx}} = 0,$$

with $V(T,x,y) = U(x)$, and where

- $\lambda(y) = \frac{\mu(y)}{\sigma(y)}$ is the Sharpe Ratio,
- and

$$\mathcal{L}_0 = \frac{1}{2}a(y)^2\frac{\partial^2}{\partial y^2} + b(y)\frac{\partial}{\partial y},$$

so $\frac{1}{\varepsilon}\mathcal{L}_0$ is the generator of Y.

Asymptotic Expansion in ε

- Look for an expansion (asymptotic as $\varepsilon \downarrow 0$):

$$V(t,x,y) = v^{(0)}(t,x,y) + \sqrt{\varepsilon}\, v^{(1)}(t,x,y) + \varepsilon v^{(2)}(t,x,y) + \cdots .$$

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- At highest order ε^{-1},

$$\mathcal{L}_0 v^{(0)} - \frac{1}{2}\rho^2 a(y)^2 \frac{(v^{(0)}_{xy})^2}{v^{(0)}_{xx}} = 0,$$

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- At next order $\varepsilon^{-1/2}$,

$$\mathcal{L}_0 v^{(1)} - \rho\lambda(y) a(y) \frac{\overbrace{v^{(0)}_{xy}}^{0} v^{(0)}_{x}}{v^{(0)}_{xx}} = 0.$$

Again, we choose $v^{(1)} = v^{(1)}(t,x)$, independent of y, which satisfies $\mathcal{L}_0 v^{(1)} = 0$.

Principal Term $v^{(0)}$

- At order one:

$$v_t^{(0)} + \mathcal{L}_0 v^{(2)} - \frac{1}{2}\lambda(y)^2 \frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = 0.$$

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- This is a Poisson equation for $v^{(2)}$ whose solvability condition (Fredholm alternative) requires that

$$v_t^{(0)} - \frac{1}{2}\bar{\lambda}^2 \frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = 0, \qquad v^{(0)}(T,x) = U(x).$$

  where $\bar{\lambda}^2$ is the square-averaged Sharpe ratio: $\bar{\lambda}^2 = \left\langle \frac{\mu^2}{\sigma^2} \right\rangle$.
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  The limit problem is constant Sharpe ratio Merton.
- The averaging result is that the effective parameter for the value function in the limit is $\bar{\lambda}^2$. When μ is constant, stochastic volatility is *harmonically averaged*:

$$\bar{\lambda}^2 = \frac{\mu}{\sigma_\star^2}, \qquad \frac{1}{\sigma_\star^2} = \left\langle \frac{1}{\sigma^2} \right\rangle.$$

Principal Linear Operator

- Introduce the (0th order) risk-tolerance function:

$$R^{(0)}(t,x) = -\frac{v_x^{(0)}(t,x)}{v_{xx}^{(0)}(t,x)},$$

and the differential operators

$$D_k = R^{(0)}(t,x)^k \frac{\partial^k}{\partial x^k}, \qquad k = 1, 2, \cdots .$$

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- Can write nonlinear term as a “diffusion”:

$$\frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = \left(\frac{v_x^{(0)}}{v_{xx}^{(0)}}\right)^2 v_{xx}^{(0)} = (R^{(0)})^2 v_{xx}^{(0)} = D_2 v^{(0)},$$

or as a “drift”:

$$\frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = \left(\frac{v_x^{(0)}}{v_{xx}^{(0)}}\right) v_x^{(0)} = -R^{(0)} v_x^{(0)} = -D_1 v^{(0)}.$$

Principal Linear Operator

- Convenient to write it as the drift diffusion combination

$$-\frac{1}{2}\bar{\lambda}^2\frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = \left(\frac{1}{2}\bar{\lambda}^2 D_2 + \bar{\lambda}^2 D_1\right) v^{(0)}.$$

- Introducing,

$$\mathcal{L}_{t,x} = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\lambda}^2 D_2 + \bar{\lambda}^2 D_1,$$

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- Next order $\varepsilon^{1/2}$:

$$\mathcal{L}_{t,x} v^{(1)} = -BD_1 D_2 v^{(0)}, \qquad B = \frac{1}{2}\rho \left\langle \lambda(y) a(y) \phi'(y) \right\rangle,$$

$$\mathcal{L}_0 \phi = \lambda(y)^2 - \bar{\lambda}^2, \qquad \text{corrector eqn.}$$

  The terminal condition is: $v^{(1)}(T, x) = 0$. LINEAR PDE.

Solution for $v^{(1)}$

- The risk-tolerance function $R^{(0)}$ satisfies Black's (fast diffusion) equation (1968):

$$R_t^{(0)} + \frac{1}{2}\bar{\lambda}^2 (R^{(0)})^2 R_{xx}^{(0)} = 0.$$

- Consequently, the operators $\mathcal{L}_{t,x}$ and D_1 acting on smooth functions of (t, x) commute: $\mathcal{L}_{t,x} D_1 = D_1 \mathcal{L}_{t,x}$.
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- Therefore $v^{(1)}$ is given by

$$v^{(1)}(t,x) = (T-t) B D_1 D_2 v^{(0)}(t,x).$$

- This is as with option pricing asymptotics, even with the (t,x)-dependent coefficients $(R^{(0)})^k$, but those are coming from solutions to Black's equation.

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$$v^{(0)} = c\frac{x^{1-\gamma}}{1-\gamma}g(t), \qquad g(t) = \exp\left(\frac{1}{2}\bar{\lambda}^2\left(\frac{1-\gamma}{\gamma}\right)(T-t)\right).$$

$$v^{(1)} = (T-t)BD_1D_2v^{(0)} = -(T-t)B\left(\frac{1-\gamma}{\gamma}\right)^2 v^{(0)}.$$

Example II: Mixture of Power Utilities

$$U(x) = c_1 \frac{x^{1-\gamma_1}}{1-\gamma_1} + c_2 \frac{x^{1-\gamma_2}}{1-\gamma_2}, \quad c_1, c_2 \geq 0, \quad \gamma_1 > \gamma_2 > 0, \quad \gamma_{1,2} \neq 1.$$

- Arrow-Pratt measure of relative risk aversion:

$$\mathrm{AP}[U] = \frac{c_1 \gamma_1 x^{-(\gamma_1-\gamma_2)} + c_2 \gamma_2}{c_1 x^{-(\gamma_1-\gamma_2)} + c_2}.$$

- Asymptotic elasticity:

$$\mathrm{AE}[U] = \lim_{x \to \infty} \left[\frac{c_1 x^{-(\gamma_1-\gamma_2)} + c_2}{\left(\frac{c_1}{1-\gamma_1}\right) x^{-(\gamma_1-\gamma_2)} + \left(\frac{c_2}{1-\gamma_2}\right)} \right] = 1 - \gamma_2 < 1.$$

- Risk-tolerance function:

$$R^{(0)} \mid_{t=T} = \left(\frac{c_1 x^{-(\gamma_1-\gamma_2)} + c_2}{c_1 \gamma_1 x^{-(\gamma_1-\gamma_2)} + c_2 \gamma_2} \right) x \sim \begin{cases} \frac{1}{\gamma_2} x & \text{as } x \to \infty \\ \frac{1}{\gamma_1} x & \text{as } x \to 0. \end{cases}$$

Mixture I: $\gamma_1 = 1.2$ and $\gamma_2 = 0.25$, $c_1 = c_2 = 0.5$

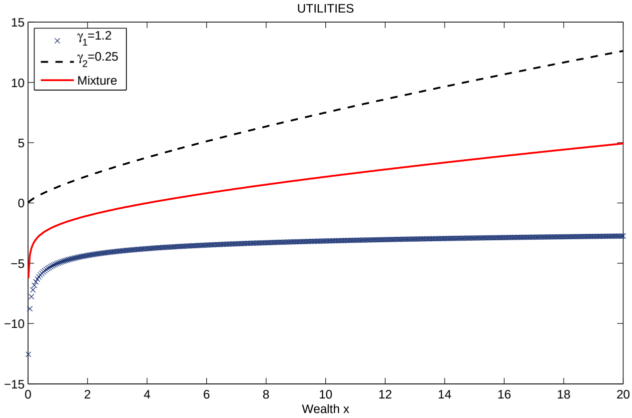

Figure: Utility Functions

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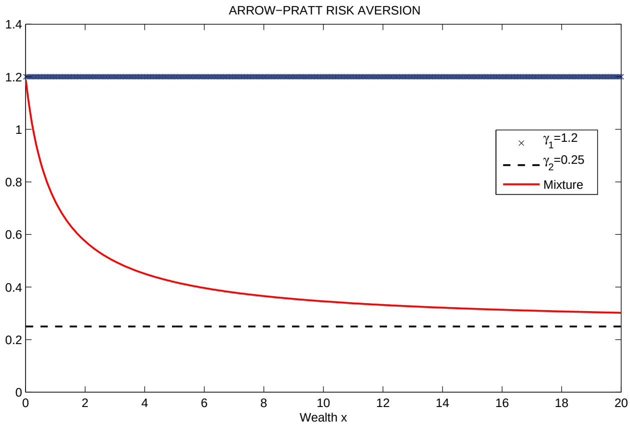

Figure: Arrow-Pratt $-xU''/U'$

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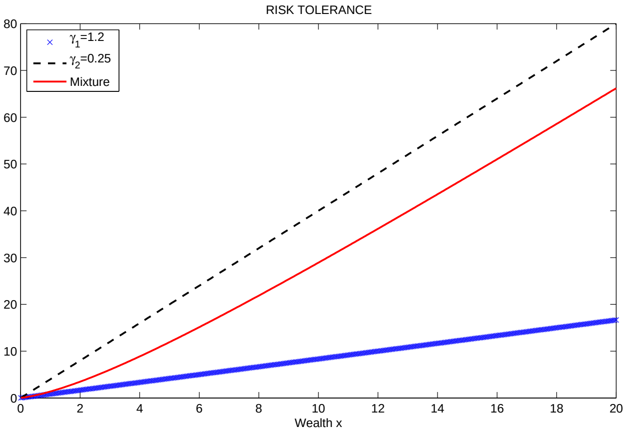

Figure: Risk-Tolerances $-U'/U''$

Mixture II: $\gamma_1 = 0.85$ and $\gamma_2 = 0.15$, $c_1 = c_2 = 0.5$

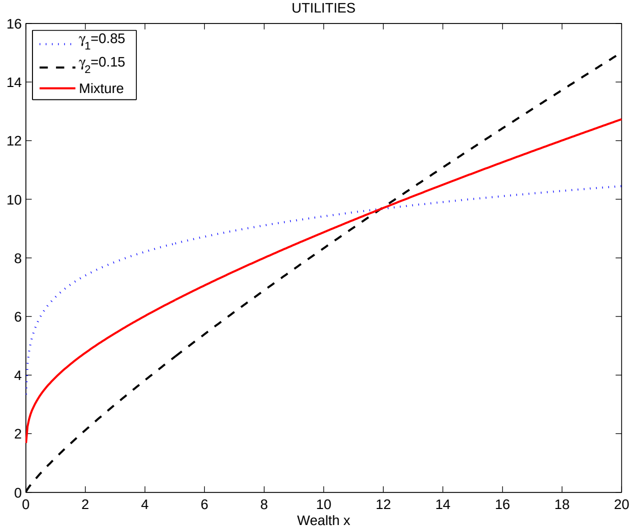

Figure: Utility Functions

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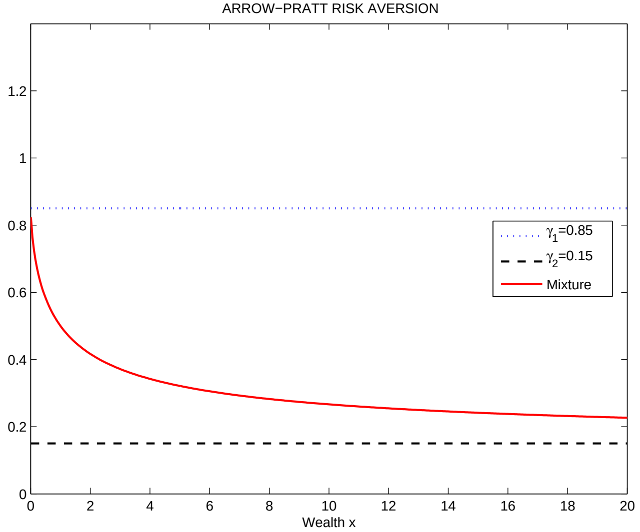

Figure: Arrow-Pratt $-xU''/U'$

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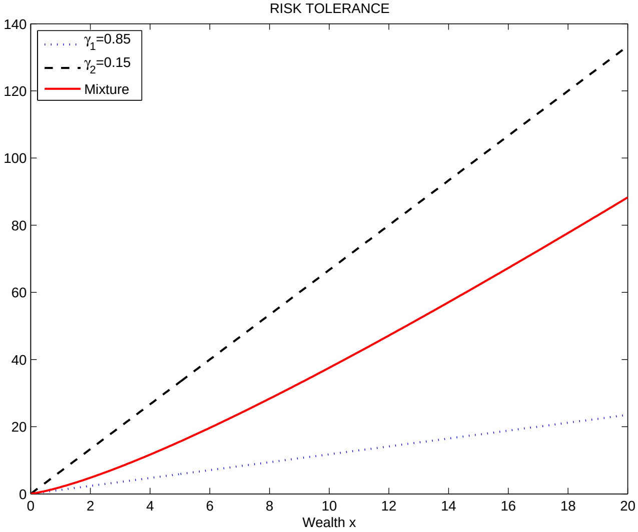

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Numerical Solution of zeroth order Merton Problem

- Discretization of the Merton PDE for $v^{(0)}(t,x)$ on $[0, x_{\max}]$

$$v_t^{(0)} - \frac{1}{2}\bar{\lambda}^2 \frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = 0, \qquad v^{(0)}(T,x) = U(x)$$

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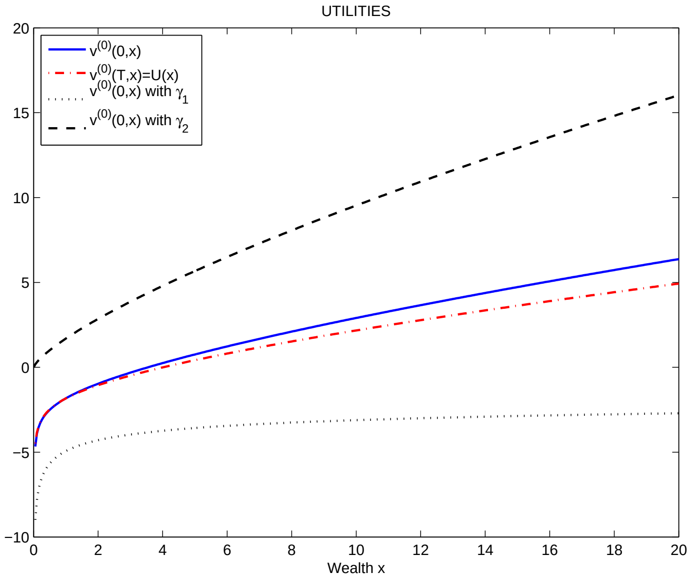
- For mixture power utilities, $R^{(0)}(t,0) = 0$, $R_x^{(0)}(t,x_{\max}) = \frac{1}{\gamma_2}$, and we know behavior of $v^{(0)}$ for large x.
- Recall asymptotic theory just needs $v^{(0)}$ and stochastic volatility effects are gotten from its derivatives ("Greeks").

Mixture I: Value function



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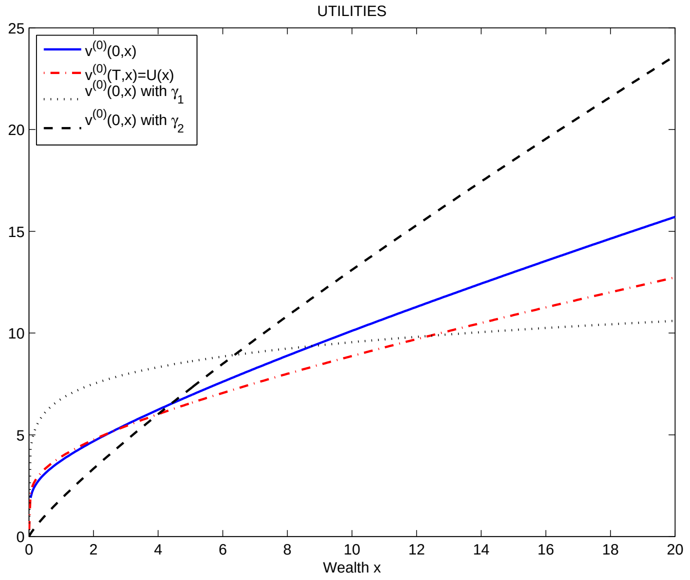

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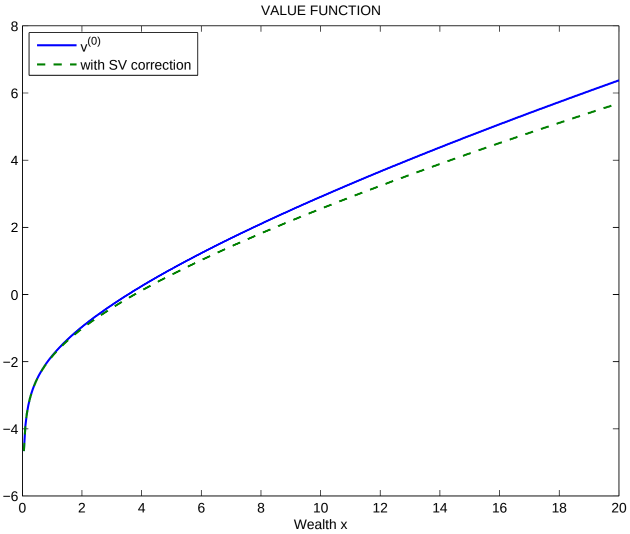

Figure: Here we plot the correction with $\sqrt{\varepsilon}\, B = 0.01$.

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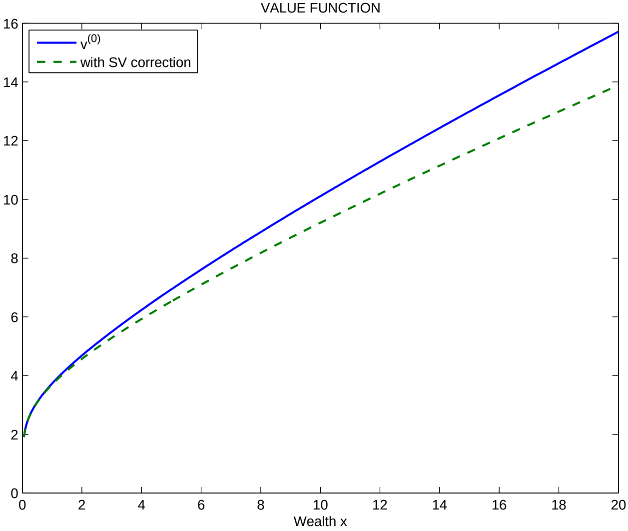

Figure: Here we plot the correction with $\sqrt{\varepsilon}B = 0.005$.

Optimal Portfolios

- Fast Volatility: $\pi^* \approx \widetilde{\pi^\varepsilon}$, where

$$\widetilde{\pi^\varepsilon} = \pi^{(0)} + \frac{\rho\sqrt{\varepsilon}}{2\sigma(y)v_x^{(0)}}\left\{B\lambda(y)(T-t)\,(D_1+D_2) + a(y)\phi'(y)\right\}D_1 D_2 v^{(0)}.$$

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$$\pi^{(0)}(t,x,y) \quad := \quad \frac{\lambda(y)}{\sigma(y)}R^{(0)}(t,x;\overline{\lambda}),$$

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- Using the “Hybrid Merton” 0th order suboptimal strategy results in the optimal value up to first order ($\sqrt{\varepsilon}$), and so the corrections to the strategy impact the value function at the $v^{(2)}$ term (order ε).

Practical (or "lazy") Merton

- For the fast volatility problem, we propose a practical strategy whose principal terms do not depend on tracking the fast moving volatility factor Y. Here, take $\mu = \text{constant}$.
- We look for an expansion of the value function

$$\bar{V}(t,x,y) = \bar{v}^{(0)}(t,x,y) + \sqrt{\varepsilon}\,\bar{v}^{(1)}(t,x,y) + \varepsilon \bar{v}^{(2)}(t,x,y) + \cdots ,$$

and of the controls:

$$\bar{\pi} = \bar{\pi}^{(0)}(t,x) + \sqrt{\varepsilon}\,\bar{\pi}^{(1)}(t,x) + \cdots ,$$

where the principal terms do not depend on y.
- At order one,

$$\bar{v}_t^{(0)} + \mathcal{L}_0 \bar{v}^{(2)} + \max_{\bar{\pi}^{(0)}} \left(\frac{1}{2}\sigma(y)^2 (\bar{\pi}^{(0)})^2 \bar{v}_{xx}^{(0)} + \bar{\pi}^{(0)} \mu \bar{v}_x^{(0)} \right) = 0.$$

For the maximizer $\bar{\pi}^{(0)}$ to not depend on y, the quantity being maximized must be y-independent.

Portfolio Optimization without Volatility Tracking

- Choose $\bar{v}^{(2)}$ to solve

$$\mathcal{L}_0\bar{v}^{(2)} + \left(\bar{v}_t^{(0)} + \frac{1}{2}\sigma(y)^2(\bar{\pi}^{(0)})^2\bar{v}_{xx}^{(0)} + \bar{\pi}^{(0)}\mu\bar{v}_x^{(0)}\right) - \left(\bar{v}_t^{(0)} + \frac{1}{2}\bar{\sigma}^2(\bar{\pi}^{(0)})^2\bar{v}_{xx}^{(0)} + \bar{\pi}^{(0)}\mu\bar{v}_x^{(0)}\right) = 0,$$

  where $\bar{\sigma}^2 = \langle\sigma(\cdot)^2\rangle$. Then

$$\bar{v}_t^{(0)} - \frac{1}{2}\frac{\mu^2}{\bar{\sigma}^2}\left(\frac{(\bar{v}_x^{(0)})^2}{\bar{v}_{xx}^{(0)}}\right) = 0, \qquad \bar{v}^{(0)}(T,x) = U(x).$$

  Merton with average volatility $\bar{\sigma}$.

- We also have:

$$\bar{\pi}^{(0)}(t,x) = -\frac{\mu}{\bar{\sigma}^2}\frac{\bar{v}_x^{(0)}}{\bar{v}_{xx}^{(0)}} = \frac{\mu}{\bar{\sigma}}R^{(0)}(t,x;\bar{\lambda}).$$

  The principal Y-independent strategy is Merton with average volatility $\bar{\sigma}$, NOT $\sigma_\star$. More conservative. Suboptimality is quantified to principal order by $|v^{(0)} - \bar{v}^{(0)}|$.

Next Order

- Choosing $\bar{v}^{(3)}$ to allow $\bar{\pi}^{(1)}$ to not depend on y leads to

$$\sqrt{\varepsilon}\,\bar{v}^{(1)} = -(T-t)\left(\frac{\mu}{\bar{\sigma}^2}\right)^3 V_3^\varepsilon D_1 D_2 \bar{v}^{(0)},$$

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  where V_3^{ε} was exactly what showed up in the option pricing asymptotics.
- But V_3^{ε} can be calibrated from the implied vol skew:

$$I \approx a\frac{\log(K/S)}{(T-t)} + b, \qquad V_3^{\varepsilon} = \frac{a}{\bar{\sigma}^3}.$$

- So the sub-optimality of the practical strategy is governed by the vol-stock correlation seen in the implied volatility skew slope.
- Can use the market implied volatility skew to gauge the performance of following the practical Merton strategy.

Performance

- We looked at relative disutility of the practical strategy:

$$RD = 1 - \frac{V^{\varepsilon}(\bar{\sigma})}{V^{\varepsilon}(\sigma_{\star})}$$

which depends on $\sigma_{\star}/\bar{\sigma}$ and V_3^{ε} (skew of implied vol).

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Dates	$\bar{\sigma}$	$\sigma_{\star}$	$\sigma_{\star}/\bar{\sigma}$
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- Conclusion: **conservatism of the practical strategy is of greater benefit in turbulent times**

Slow Scale Volatility Asymptotics

- Now suppose stochastic volatility is slowly fluctuating:

$$
\begin{aligned}
dS_t &= \mu(Z_t)S_t\,dt + \sigma(Z_t)S_t\,dW_t^{(1)} \\
dZ_t &= \delta c(Z_t)\,dt + \sqrt{\delta}g(Z_t)\left(\rho\,dW_t^{(1)} + \sqrt{1-\rho^2}\,dW_t^{(2)}\right),
\end{aligned}
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  where δ is the small parameter for expansion:

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V(t,x,z) = v^{(0)}(t,x,z) + \sqrt{\delta}\,v^{(1)}(t,x,z) + \delta v^{(2)}(t,x,z) + \cdots .
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  Then $v^{(0)}$ is the Merton value function with frozen Sharpe ratio $\lambda(z)$. Let $R^{(0)}(t,x,z) = -v_x^{(0)}(t,x,z)/v_{xx}^{(0)}(t,x,z)$.

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\mathcal{L}_{t,x} = \frac{\partial}{\partial t} + \frac{1}{2}\lambda(z)^2 D_2 + \lambda(z)^2 D_1, \quad D_k = (R^{(0)})^k \frac{\partial^k}{\partial x^k}
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  where now $\mathcal{L}_{t,x}$ has coefficients depending on z.
- Then $\mathcal{L}_{t,x}v^{(0)} = 0$ with $v^{(0)}(T,x,z) = U(x)$.
- Taking the order $\sqrt{\delta}$ terms leads to

$$\mathcal{L}_{t,x}v^{(1)} = -\rho\lambda(z)g(z)D_1 v_z^{(0)}, \qquad v^{(1)}(T,x,z) = 0.$$

Vega-Gamma Relationship

- In European option pricing under constant volatility :

$$\frac{\partial C_{BS}}{\partial \sigma} = (T - t)\sigma S^2 \frac{\partial^2 C_{BS}}{\partial S^2}.$$

  Long convexity $\Rightarrow$ long volatility .
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- Therefore

$$\mathcal{L}_{t,x} v^{(1)} = -\rho\lambda(z)g(z)D_1 v_z^{(0)} = \rho\lambda(z)g(z)D_1\left((T-t)\lambda\lambda' D_2 v^{(0)}\right)$$

$$\Rightarrow \quad v^{(1)}(t,x,z) = \frac{1}{2}(T-t)\rho\lambda(z)g(z)D_1 v_z^{(0)}.$$

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THANK YOU FOR YOUR ATTENTION