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# ARBITRAGES AND DRIFT INFORMATION

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FEDERATION BANCAIRE FRANCAISE

Time present and time past

Are both perhaps present in time future,

And time future contained in time past

T.S. Eliot, The four quartets

The goal of this talk is to show, with some examples, how arbitrages can occur using an extra-information (in a progressive enlargement framework), mainly information on a non-traded default event. In a log-maximization framework, the impact of the new information is contained in the drift of the prices, written as semi-martingales in the new filtration.

1. Toy Example
2. Brownian filtration and progressive enlargement
3. Arbitrages with honest times

Toy Example

## The Model

We consider the case where the sources of randomness are the occurrence of two random times $\tau_1$ and $\tau_2$ (finite positive random variables).

We denote by $\mathbb{F}$ the filtration generated by the process $(H_t := \mathbb{1}_{\tau_1 \leq t})$. This will represent the information available to all market participants which are using $\mathbb{F}$-predictable strategies, when dealing with $\mathbb{F}$-adapted prices.

We denote by $\mathbb{H}^2$ the filtration generated by the process $(H_t^2 := \mathbb{1}_{\tau_2 \leq t})$ and by $\mathbb{G}$ the filtration generated by both processes $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^2$. One participant (called "informed") has access to the information $\mathbb{H}^2$, i.e., he is able to use $\mathbb{G}$-predictable strategies when dealing with $\mathbb{F}$-adapted prices.

We denote by $G(t,s) = \mathbb{P}(\tau_1 > t, \tau_2 > s)$ the survival probability (a deterministic function) of the pair $(\tau_1, \tau_2)$ assumed to be strictly positive and continuously differentiable in both variables. Note that $G(t,0) = \mathbb{P}(\tau_1 > t)$ is the survival probability of $\tau_1$. The function $G$ is known from all market participants.

Assuming that $\mathbb{P}$ is the pricing measure and that the interest rate is null,

$$P(t,T) \quad := \quad \mathbb{P}(\tau_1 > T | \mathcal{F}_t) = 1\!\!1_{t \le \tau_1} \frac{G(T,0)}{G(t,0)}$$

represents the price of a defaultable bond.

**Log-utility maximization**

We assume, for simplicity, that $\tau_1$ has an exponential law of parameter $\lambda$.

We introduce the fundamental $(\mathbb{P}, \mathbb{F})$ martingale

$$M_t^1 := H_t - \lambda(t \wedge \tau_1) = H_t - \int_0^t (1 - H_s)\lambda ds.$$

Then,

$$\begin{aligned} P(t,T) &= \mathbb{1}_{t \leq \tau_1} \frac{G(T,0)}{G(t,0)} = \mathbb{1}_{t \leq \tau_1} e^{-\lambda(T-t)} \\ dP(t,T) &= -e^{-\lambda(T-t)} dM_t^1 \end{aligned}$$

We now compute the **growth optimal portfolio** in the two cases where the asset $P(\cdot, T)$ is traded and the agent has (or has not) information on $\tau_2$.

More precisely, we solve the log utility maximization

$$\sup \mathbb{E}(\ln(X_T^{x,\pi})) \ \text{ for } \ \pi \in \mathbb{F}, \qquad \text{and } \ \text{for } \ \pi \in \mathbb{G}$$

where

$$dXx, \pi_t = \vartheta_t dP(t,T) = \pi_t X_{t-} dM_t^1 = \pi_t X_{t-}^{x,\pi}(dH_t - (1-H_t)\lambda dt)$$

or

$$X_t^{x,\pi} = x \exp(-\lambda \int_0^{t\wedge\tau_1} \pi_s ds)(1+\pi_{\tau_1})^{H_t}$$

so that

$$\ln(X_t^{x,\pi}) = \ln x + \int_0^t \ln(1+\pi_s) dH_s - \lambda \int_0^{t\wedge\tau_1} \pi_s ds$$

• Using $\mathbb{F}$-predictable strategies

$$\sup\{\mathbb{E}(\ln(X_T^{x,\pi})), \pi \in \mathbb{F}\} = \ln x$$

• Using $\mathbb{G}$-predictable strategies, we have to note that $M^1$ is not a $\mathbb{G}$-martingale, and we have to deal with the $(\mathbb{P}, \mathbb{G})$-martingale

$$M_t^2 := H_t - \int_0^t (1 - H_s)\lambda_s^2 ds,$$

where

$$\lambda_t^2 = 1\!\!1_{t \leq \tau_2} \frac{-\partial_1 G(t,t)}{G(t,t)} + 1\!\!1_{\tau_2 < t} \frac{\partial_{12} G(t,\tau_2)}{-\partial_1 G(t,\tau_2)}$$

$$\begin{aligned} \ln(X_t^{x,\pi}) &= \ln x + \int_0^t \ln(1+\pi_s) dH_s - \lambda \int_0^{t \wedge \tau_1} \pi_s ds \\ &= \ln x + \int_0^t \ln(1+\pi_s) dM_s^2 + \int_0^{t \wedge \tau_1} \ln(1+\pi_s)\lambda_s^2 ds - \lambda \int_0^{t \wedge \tau_1} \pi_s ds \end{aligned}$$

the optimal strategy is $\pi$ such that $1+\pi_t = \frac{\lambda_t^2}{\lambda}$. Then

$$\sup\{\mathbb{E}(\ln(X_T^{x,\pi}), \pi \in \mathbb{G}\} = \ln x + \mathbb{E}\left(\int_0^{T\wedge\tau_1}\left(\lambda_s^2 \ln \frac{\lambda_s^2}{\lambda} - (\lambda_s^2 - \lambda)\right) ds\right) \geq \ln x$$

and, under adequate hypotheses $\mathbb{E}\left(\int_0^{T\wedge\tau_1}\left(\lambda_s^2 \ln \frac{\lambda_s^2}{\lambda} - (\lambda_s^2 - \lambda)\right) ds\right) < \infty$

Of course, it may happen that

$$\mathbb{E}\left(\int_0^{T\wedge\tau_1}\left(\lambda_s^2\ln\frac{\lambda_s^2}{\lambda}-(\lambda_s^2-\lambda)\right)ds\right)=0$$

(e.g. if $\tau_2$ is independent from $\tau_1$).

It can also happen that

$$\mathbb{E}\left(\int_0^{T\wedge\tau_1}\left(\lambda_s^2\ln\frac{\lambda_s^2}{\lambda}-(\lambda_s^2-\lambda)\right)ds\right)=\infty$$

e.g. if $\tau_1=\tau_2+\epsilon$.

Drift information in a Progressive Enlargement in a Brownian setting

We assume in this part that

- $\mathbb{F}$ is the filtration generated by a Brownian motion $W$ and $\mathbb{G}$ is a filtration larger than $\mathbb{F}$
- there exists an integrable $\mathbb{G}$-adapted process $\mu^{\mathbb{G}}$ such that $dW_t = dW_t^{\mathbb{G}} + \mu_t^{\mathbb{G}} dt$ where $W^{\mathbb{G}}$ is a $\mathbb{G}$-BM,

- the financial market where a risky asset with price $S$ (an $\mathbb{F}$-adapted positive process) and a riskless asset $S^0 \equiv 1$ are traded is arbitrage free. More precisely, we assume w.l.g. that $S$ is a $(\mathbb{P}, \mathbb{F})$ (local) martingale, $dS_t = S_t \sigma_t dW_t$.

Let $X$ be the wealth process associated with a $\mathbb{G}$-predictable strategy

$$dX_t = \widehat{\pi}_t dS_t = \widehat{\pi}_t S_t \sigma_t dW_t = \pi_t X_t dW_t = \pi_t X_t (dW_t^{\mathbb{G}} + \mu_t^{\mathbb{G}} dt)$$

so that

$$X_t = x \exp\left( \int_0^t \pi_s dW_s^{\mathbb{G}} - \frac{1}{2} \int_0^t \pi_s^2 ds + \int_0^t \pi_s \mu_s^{\mathbb{G}} ds \right)$$

It is then easy to see that the optimal $\pi$ is $\pi^* = \mu^{\mathbb{G}}$ and that

$$\ln X_t^* = \ln x + \int_0^t \pi_s^* dW_s^{\mathbb{G}} + \frac{1}{2} \int_0^t (\mu_s^{\mathbb{G}})^2 ds$$

so that

$$\sup_{\pi \in \mathbb{F}} \mathbb{E}(\ln X_T) = \ln x < \sup_{\pi \in \mathbb{G}} \mathbb{E}(\ln X_T) = \ln x + \mathbb{E}\left( \frac{1}{2} \int_0^t (\mu_s^{\mathbb{G}})^2 ds \right)$$

which leads to a finite utility if

$$\mathbb{E}\left( \int_0^t (\mu_s^{\mathbb{G}})^2 ds \right) < \infty$$

From $dS_t = S_t\sigma_t dW_t = S_t\sigma_t(dW_t^{\mathbb{G}} + \mu_t^{\mathbb{G}} dt)$, we see that if $L_t := \mathcal{E}(-\mu^{\mathbb{G}} W^{\mathbb{G}})_t$ is a $\mathbb{G}$-martingale, NFLVR holds, and if $L$ is a local martingale, one say that the ***no arbitrages of the first kind*** *condition* (NA1) holds (there exists a positive local martingale $L$ such that $SL$ is a $\mathbb{P}$ local martingale). The local martingale $L$ is then called a deflator.

# Particular cases of enlargement of filtration

## Density hypothesis

We assume that there exists a **positive** $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(\omega, u) \to \alpha_t(\omega, u)$ which satisfies for any Borel bounded function $f$,

$$\mathbb{E}(f(\tau)|\mathcal{F}_t) = \int_{\mathbb{R}_+} f(u)\alpha_t(u)\nu(du), \quad \mathbb{P} - a.s.$$

where $\nu$ is the law of $\tau$.

Under the positive density hypothesis, it can be proved that the probability $\mathbb{P}^*$, defined on $\mathbb{F}^\tau = \mathbb{F} \vee \sigma(\tau)$ as

$$d\mathbb{P}^*|_{\mathcal{F}_t \vee \sigma(\tau)} = \frac{1}{\alpha_t(\tau)} d\mathbb{P}|_{\mathcal{F}_t \vee \sigma(\tau)}$$

satisfies the following assertions

(i) Under $\mathbb{P}^*$, $\tau$ is independent from $\mathcal{F}_t$ for any $t$

(ii) $\mathbb{P}^*|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$

(iii) $\mathbb{P}^*|_{\sigma(\tau)} = \mathbb{P}|_{\sigma(\tau)}$

The $(\mathbb{F}, \mathbb{P})$ martingale $S$ is - using the independence property - an $(\mathbb{F}^\tau, \mathbb{P}^*)$ martingale and $\mathbb{P}^*$ is an e.m.m. In that case,

$$W_t = W_t^{\mathbb{F}^\tau} + \int_0^t \frac{d\langle W, \alpha(u)\rangle_s}{\alpha_s(u)}|_{u=\tau}$$

The process $\alpha(u)$ being an $\mathbb{F}$ martingale, $d\alpha_t(u) = \sigma_t(u) dW_t$ and it follows that

$$\mu_s^{\mathbb{F}^\tau} = \frac{\sigma_s(\tau)}{\alpha_s(\tau)}$$

In that model NFLVR as well as NA1 hold in the filtration $\mathbb{F}^\tau$.

If $\mathbb{G}$ is the smallest filtration that contains $\mathbb{F}$ and makes $\tau$ a stopping time,

$$W_t = W_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{d\langle W, Z\rangle_s}{Z_s} + \int_{t\wedge\tau}^{t} \frac{d\langle W, \alpha(u)\rangle_s}{\alpha_s(u)}\Big|_{u=\tau}$$

where

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^{\infty} \alpha_t(u) f(u) du$$

and $W^{\mathbb{G}}$ is a $\mathbb{G}$-Brownian motion.

It follows that

$$\mu_s^{\mathbb{G}} = \mathbb{1}_{s\leq\tau} \frac{1}{Z_s} \int_s^{\infty} \sigma_s(u)\nu(du) + \mathbb{1}_{\tau<s} \frac{\sigma_s(\tau)}{\alpha_s(\tau)}$$

**Immersion setting**

We recall that the filtration $\mathbb{F}$ is immersed in $\mathbb{G}$ if any $\mathbb{F}$ martingale is a $\mathbb{G}$ martingale. This is equivalent to

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$$

This is the case in many default models (e.g., based on a Cox process construction).

Let $S$ be an $\mathbb{F}$ local martingale, then it is a $\mathbb{G}$ local martingale as well.

Under immersion, the optimal log utility (in fact any utility maximisation) is the same in the two filtrations.

**Emery's Example**

We present an example where there are no arbitrages before $\tau$ and arbitrages after $\tau$ (roughly speaking, $\mathbb{E}(\int_0^\tau \mu_s^2 ds) = 0, \mathbb{E}(\int_\tau^T \mu_s^2 ds) = \infty$)

Let $S$ be defined through $dS_t = \sigma S_t dW_t$, where $W$ is a Brownian motion.

Let $\tau = \sup\{t \leq 1 \,:\, S_1 - 2S_t = 0\}$, that is the last time before 1 when the price is equal to half of its terminal value at time 1.

Note that

$$\{\tau \le t\} = \{\inf_{t \le s \le 1} 2\frac{S_s}{S_t} \ge \frac{S_1}{S_t}\}$$

therefore

$$\mathbb{P}(\tau \le t|\mathcal{F}_t) = \mathbb{P}(\inf_{t \le s \le 1} 2S_{s-t} \ge S_{1-t}) = \Phi(1-t)$$

where $\Phi(u) = \mathbb{P}(\inf_{s \le u} 2S_s \ge S_u)$. It follows that the Azéma supermartingale is a deterministic decreasing function, hence, $\tau$ **is a pseudo-stopping time,** hence $S$ is a $\mathbb{G}$-martingale up to time $\tau$ and there are no arbitrages up to $\tau$. The information drift is null.

There are obviously classical arbitrages after $\tau$, since, at time $\tau$, one knows the value of $S_1$ and $S_1 > S_\tau$.

*Let $\mathbb{K} = (\mathbb{F}, \mathbb{G})$ be a filtration. A non-negative $\mathcal{K}_\infty$-measurable random variable $\xi$ with $\mathbb{P}(\xi > 0) > 0$ yields an arbitrage of the fist kind if for all $x > 0$ there exists an element $\theta^x \in \mathcal{A}_x^{\mathbb{K}}$ such that $V(x, \theta^x)_\infty := x + (\theta^x \centerdot S)_\infty \geq \xi$ $\mathbb{P}$-a.s.*

The random variable $\xi := \frac{1}{2} S_1$ is an arbitrage of the first kind. Indeed, for $t > \tau$, and $x > 0$, one has, for $\theta^x = 1$

$$x + \int_\tau^1 \theta_s^x dS_s = x + S_1 - S_\tau = x + \xi > \xi$$

One can check that indeed $\mathbb{E}(\int_\tau^1 \mu_s^2 ds) = \infty$.

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Let $\mathbb{K} = (\mathbb{F}, \mathbb{G})$ be a filtration. A non-negative $\mathcal{K}_\infty$-measurable random variable $\xi$ with $\mathbb{P}\left(\xi > 0\right) > 0$ yields **an arbitrage of the fist kind** if for all $x > 0$ there exists an element $\theta^x \in \mathcal{A}_x^{\mathbb{K}}$ such that $V\left(x, \theta^x\right)_\infty := x + \left(\theta^x \centerdot S\right)_\infty \geq \xi$ $\mathbb{P}$-a.s.

The random variable $\xi := \frac{1}{2} S_1$ is an arbitrage of the first kind in $\mathbb{G}$. Indeed, for $t > \tau$, and $x > 0$, one has, for $\theta^x = 1$

$$x + \int_\tau^1 \theta_s^x dS_s = x + S_1 - S_\tau = x + \xi > \xi$$

One can check that indeed $\mathbb{E}(\int_\tau^1 \mu_s^2 ds) = \infty$.

# Enlargement of filtration results

We restrict our attention to the case where $\mathbb{F}$ is a Brownian filtration and $\tau$ avoids $\mathbb{F}$-stopping times.

We define the (continuous) $\mathbb{F}$-supermartingale

$$Z_t := \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right).$$

One can write the Doob-Meyer decomposition of $Z$ as

$$Z = m - A^p$$

where $m$ is an $\mathbb{F}$-martingale and $A^p$ is a (predictable) increasing process

**Note that $m$ is non-negative**: indeed $m_t = \mathbb{E}(A^p_\infty | \mathcal{F}_t)$.

## Arbitrages of the first kind

# Before $\tau$

To any $\mathbb{F}$ local martingale $X$, we associate the $\mathbb{G}$ local martingale $\widehat{X}$ (stopped at time $\tau$) defined as

$$\widehat{X}_t := X_t^\tau - \int_0^{t\wedge\tau} \frac{1}{Z_s} d\langle X, m\rangle_s$$

In particular, in a Brownian filtration

$$\widehat{W}_t := W_t^\tau - \int_0^{t\wedge\tau} \frac{1}{Z_s} d\langle W, m\rangle_s = W_t^\tau - \int_0^{t\wedge\tau} \mu_s^{\mathbb{G}} ds$$

hence, NA1 holds before $\tau$.

More generally, if $\mathbb{F}$ is a filtration such that all martingales are continuous and $\tau$ avoids stopping times, **NA1 holds before $\tau$.**

Let $\widehat{m}$ be the $\mathbb{G}$-martingale stopped at time $\tau$ associated with $m$, on $t \leq \tau$

$$\widehat{m}_t := m_t^\tau - \int_0^t \frac{d\langle m, m\rangle_s^{\mathbb{F}}}{Z_s}$$

and define a positive $\mathbb{G}$ local martingale $L$ as $dL_t = -\frac{L_t}{Z_t} d\widehat{m}_t$. Recall that

$$\widehat{S}_t := S_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle S, m\rangle_s^{\mathbb{F}}}{Z_s}$$

is a $\mathbb{G}$ local martingale. From integration by parts, we obtain

$$\begin{aligned} d(LS^\tau)_t &= L_t dS_t^\tau + S_t dL_t + d\langle L, S^\tau\rangle_t^{\mathbb{G}} \\ &\overset{\mathbb{G}-\text{mart}}{=} L_t \frac{1}{Z_t} d\langle S, m\rangle_t^{\mathbb{F}} + \frac{1}{Z_{t-}} L_{t-} d\langle S, \widehat{m}\rangle_t^{\mathbb{G}} \\ &\overset{\mathbb{G}-\text{mart}}{=} L_t \frac{1}{Z_t} \left(d\langle S, m\rangle_t - d\langle S, m\rangle_t\right) = 0 \end{aligned}$$

Since $SL$ is a $\mathbb{G}$-local martingale, NA1 holds .

More generally, if $\mathbb{F}$ is a filtration such that all martingales are continuous and $\tau$ avoids stopping times, **NA holds before $\tau$.**

Let $\widehat{m}$ be the $\mathbb{G}$-martingale stopped at time $\tau$ associated with $m$, on $t \leq \tau$

$$\widehat{m}_t := m_t^\tau - \int_0^t \frac{d\langle m, m\rangle_s^{\mathbb{F}}}{Z_s}$$

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is a $\mathbb{G}$ local martingale. From integration by parts, we obtain

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Since $SL$ is a $\mathbb{G}$-local martingale, NA1 holds.

# Case of a Poisson filtration

In the general case,

$$\widehat{X}_t := X_t^\tau - \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s^{\mathbb{F}}$$

holds true if $m$ is defined as $Z = m - A^{o,\mathbb{F}}$ where $A^{o,\mathbb{F}}$ is the dual OPTIONAL projection of $1\!\!1_{[\![0,\tau[\![}$. However, one faces some technical problem, as shown in the following simple example.

We assume that $S$ is an $\mathbb{F}$-martingale of the form $dS_t = S_{t-}\psi_t dM_t$, where $\psi$ is a predictable process, satisfying $\psi > -1$ and $\psi \neq 0$, where $M$ is the compensated martingale of a standard Poisson process.

From Predictable Representation Property, $dm_t = \varphi_t dM_t$ for some $\mathbb{F}$-predictable process $\varphi$, so that, on $t \leq \tau$,

$$d\widehat{m}_t = dm_t - \frac{1}{Z_{t-}} d\langle m, m\rangle_t = dm_t - \frac{1}{Z_{t-}} \lambda \varphi_t^2 dt.$$

In a Poisson setting, for any random time $\tau$ such that $\int_0^\tau 1\!\!1_{Z_{t-}+\varphi_t} dt = 0$, NA1 holds before $\tau$. Indeed

$$L = \mathcal{E}\left(-\frac{1}{Z_- + \varphi} \centerdot \widehat{m}\right) = \mathcal{E}\left(-\frac{\varphi}{Z_- + \varphi} \centerdot \widehat{M}\right),$$

is a $\mathbb{F}^\tau$-local martingale deflator for $S^\tau$.

We are looking for an $\mathbb{F}^\tau$-local martingale deflator of the form $dL_t = L_{t-}\,\kappa_t\, d\widehat{m}_t$ (and $\psi_t\kappa_t > -1$) so that $L$ is positive and $S^\tau L$ is an $\mathbb{F}^\tau$-local martingale. Integration by parts formula leads to (on $t \leq \tau$)

$$\begin{aligned}
d(LS)_t &= L_{t-}dS_t + S_{t-}dL_t + d[L,S]_t \\
&\overset{\mathbb{F}^\tau-mart}{=} L_{t-}\, S_{t-}\, \psi_t \frac{1}{Z_{t-}} d\langle M, m\rangle_t + L_{t-}\, S_{t-}\, \kappa_t\psi_t\varphi_t dN_t \\
&\overset{\mathbb{F}^\tau-mart}{=} L_{t-}\, S_{t-}\, \psi_t \frac{1}{Z_{t-}} \varphi_t \lambda dt + L_{t-}\, S_{t-}\, \kappa_t\, \psi_t\, \varphi_t\, \lambda(1 + \frac{1}{Z_{t-}}\varphi_t)dt \\
&= L_{t-}\, S_{t-}\, \psi_t\, \varphi_t\, \lambda \left(\frac{1}{Z_{t-}} + \kappa_t(1 + \frac{1}{Z_{t-}}\varphi_t)\right) dt.
\end{aligned}$$

Therefore, for $\kappa_t = -\frac{1}{Z_{t-}+\varphi_t}$, one obtains a deflator. Note that

$$dL_t = L_{t-}\kappa_t d\widehat{m}_t = -L_{t-}\frac{1}{Z_{t-}+\varphi_t}\varphi_t d\widehat{M}_t$$

is indeed a positive $\mathbb{F}^\tau$-local martingale, since $\frac{1}{Z_{t-}+\varphi_t}\varphi_t < 1$.

# General result

We introduce $\widetilde{Z}_t = \mathbb{P}(\tau \le t | \mathcal{F}_t)$. **For any (bounded) $X$ satisfying NA1$(\mathbb{F})$, $X^\tau$ satisfies NA1$(\mathbb{G})$ if and only if the thin set $\Lambda := \left\{ \widetilde{Z} = 0 \ \& \ Z_- > 0 \right\}$ is evanescent, equivalently, $\eta = \infty$)** where $\eta = \zeta 1\!\!1_{\{\widetilde{Z}_\zeta = 0 < Z_{\zeta -}\}} + \infty 1\!\!1_{\Lambda^c}$ and $\zeta := \inf\{t : Z_t = 0\}$

The proof in Aksamit et al. is based on the following (new) decomposition: if $X$ is an $\mathbb{F}$-local martingale, the process

$$X_t^{\mathbb{G}} := X_t^\tau - \int_0^{t\wedge\tau} \frac{1}{\widetilde{Z}_s} d[m, X]_s + (\Delta X_\eta \, 1\!\!1_{[\![\eta,\infty[\![})^{p,\mathbb{F}}_{t\wedge\tau}, \quad t \ge 0$$

is a $\mathbb{G}$-local martingale.

Then, one defines

$$\ell = \mathcal{E}(-\frac{1}{Z_-} 1\!\!1_{[0,\tau]} \centerdot m^{\mathbb{G}}).$$

Under the evanescent condition, $\ell$ is a local martingale deflator.

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$$X_t^{\mathbb{G}} := X_t^\tau - \int_0^{t \wedge \tau} \frac{1}{\widetilde{Z}_s} d[m, X]_s + (\Delta X_\eta \, \mathbb{1}_{[\![\eta, \infty[\![})_{t \wedge \tau}^{p,\mathbb{F}}, \quad t \ge 0$$

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Then, one defines

$$\ell = \mathcal{E}(-\frac{1}{Z_-} \mathbb{1}_{[0,\tau]} \centerdot m^{\mathbb{G}}).$$

Under the evanescent condition, $\ell$ is a local martingale deflator.

# Honest times

In order to study the behavior after $\tau$, one needs an hypothesis which implies that $W$ is a $\mathbb{G}$ semi-martingale. We restrict our attention to the case where all $\mathbb{F}$-martingales are continuous and $\tau$ avoids $\mathbb{F}$-stopping times.

In a continuous filtration, a random time $\tau$ which avoids $\mathbb{F}$-stopping times is **honest** if, $Z_\tau = 1$. This is equivalent to $A^p_t = A^p_{t\wedge\tau}$.

In the case where $\tau = \sup\{t \leq T, \, S_t = \sup_{s \leq T} S_s\}$, one can find, in Dellacherie, Maisonneuve, Meyer (1992), *Probabilités et Potentiel, chapitres XVII-XXIV: Processus de Markov (fin), Compléments de calcul stochastique*, page 137 ***Par exemple, $S_t$ peut représenter le cours d'une certaine action à l'instant $t$, et $\tau$ est le moment idéal pour vendre son paquet d'actions. Tous les spéculateurs cherchent à connaître $\tau$ sans jamais y parvenir, d'où son nom de variable aléatoire honnête.***

For instance, $S_t$ may represent the price of some stock at time $t$ and $\tau$ is the optimal time to liquidate a position in that stock. Every speculator strives to know when $\tau$ will occur, without ever achieving this goal. Hence, the name of honest random variable.

**Let $\tau$ be a finite honest time and assume that the market $(S^0, S)$ is complete. Then, if $\tau$ is not an $\mathbb{F}$-stopping time, there are classical arbitrages before and after $\tau$.**

## Before $\tau$

From $m = Z + A^p$ and $Z_\tau = 1$, we deduce that $m_\tau \geq 1$.

Since $\tau$ is not a stopping time, $\mathbb{P}(A^p_\tau > 0) > 0$.

The market being complete, the martingale $m$ is the value of a self financing portfolio, with initial value 1, and $m_\tau = 1 + \int_0^\tau \varphi_s dS_s$ for a predictable $\varphi$. Since $m_t \geq 0$, the strategy $\varphi$ is admissible.

**Classical arbitrages after $\tau$:** We restrict our attention to the case where $A^0$ is continuous (i.e., $\tau$ avoids $\mathbb{F}$ stopping times) so that $\widetilde{Z}_t = Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$

Using $m = Z + A^p$, one obtains that, for $t > \tau$, $m_t - m_\tau = Z_t - 1$.

Consider the (finite) $\mathbb{G}$-stopping time

$$\nu := \inf\{s > \tau : Z_s \leq \frac{1}{2}\}.$$

Then,

$$m_\nu - m_\tau = Z_\nu - 1 \leq \frac{-1}{2} \leq 0,$$

and, as $\tau$ is not an $\mathbb{F}$-stopping time,

$$\mathbb{P}(m_\nu - m_\tau < 0) = 1 > 0.$$

Hence $-\int_\tau^{t\wedge\nu} \varphi_s dS_s = m_{\tau\wedge t} - m_{t\wedge\nu}$ is the value of a self-financing strategy with initial value 0 and terminal value $m_\tau - m_\nu \geq 0$ satisfying $\mathbb{P}(m_\tau - m_\nu > 0) > 0$.

From $m = Z + A$ and the fact that $A_t = A_{t\wedge\tau}$, one obtains that $m_t - m_\tau = Z_t - Z_\tau \geq -1$, hence the strategy is admissible.

## Examples in a Brownian filtration

In this section, we assume that

$$S_t = \exp(\sigma W_t - \frac{1}{2}\sigma^2 t), \quad \sigma > 0 \, .$$

- Consider the following finite honest time

$$g := \sup\{t \,:\, S_t = a\},$$

where $0 < a < 1$. This time is well defined, since $S_t$ goes to 0 when $t$ goes to infinity.

Then $Z_t = 1 - (1 - \frac{S_t}{a})^+$, and

$$dZ_t = \mathbb{1}_{\{S_t < a\}} \frac{1}{a} dS_t - \frac{1}{2a} d\ell_t^a$$

Therefore,

$$\varphi := \frac{1}{a}\mathbb{1}_{\{S<a\}}$$

It follows that

$$
\begin{aligned}
W_t &= W_t^{\mathbb{G}} + \int_0^{t\wedge g} \frac{d\langle W, m\rangle_s}{Z_s} - \int_{t\wedge g}^{t} \frac{d\langle W, m\rangle_s}{1 - Z_s} \\
&= W_t^{\mathbb{G}} + \int_0^{t\wedge g} \sigma 1\!\!1_{\{S_s<a\}} ds - \int_{t\wedge g}^{t} 1\!\!1_{\{S_s<a\}} \frac{\sigma S_s}{a - S_s} ds \\
&= W_t^{\mathbb{G}} + \int_0^{t\wedge g} \sigma 1\!\!1_{\{S_s<a\}} ds - \int_{t\wedge g}^{t} \frac{\sigma S_s}{a - S_s} ds
\end{aligned}
$$

On the interval $[0, g]$, one has $\mu_s = \sigma 1\!\!1_{\{S_s<a\}}$, hence

$$
\mathbb{E}\left(\int_0^g \mu_s^2 ds\right) = \sigma^2 \mathbb{E}\left(\int_0^g 1\!\!1_{S_s<a} ds\right) < \sigma^2 \mathbb{E}(g) < \infty
$$

- Let, $S_t^* = \sup\{S_s, s \le t\}$ and

$$\tau = \sup\{t \,:\, S_t = S_\infty^*\} = \sup\{t \,:\, S_t = S_t^*\}$$

Then, $Z_t = \frac{S_t}{S_t^*}$ and $dm_t = \frac{1}{S_t^*} dS_t$, therefore $\varphi_t = \frac{1}{S_t^*}$.

$$\begin{aligned} W_t &= W_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{d\langle W, m\rangle_s}{Z_s} - \int_{t\wedge\tau}^t \frac{d\langle W, m\rangle_s}{1 - Z_s} \\ &= W_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \sigma ds - \int_{t\wedge\tau}^t \frac{\sigma S_s}{S_s^* - S_s} ds \end{aligned}$$

On the interval $[0, \tau]$, one has $\mu_s = \sigma$, hence

$$\mathbb{E}\left(\int_0^\tau \mu_s^2 ds\right) = \sigma^2 \mathbb{E}(\tau) < \infty$$

## After $\tau$

We need an hypothesis that implies that $W$ is a $\mathbb{G}$-semi martingale. We have presented the density hypothesis. Another case is the one of honest times.

We now assume that $\tau$ is a honest time (i.e., the end of a predictable set), which satisfies $Z_\tau < 1$.

If $Z_\tau = 1$, the random time $\tau$ avoids $\mathbb{F}$ stopping times and one can prove, in the case of Brownian filtration that there exists an arbitrage of the first kind.

**Assume that $\tau$ is a honest time, which satisfies $Z_\tau < 1$ and that all $\mathbb{F}$ martingales are continuous. Then, NA1 holds after $\tau$.** A deflator is given by $dL_t = -\frac{L_t}{1-Z_t} d\widehat{m}_t$.

The proof is based on Itô's calculus and the fact that, for any $\mathbb{F}$ martingale $X$ (in particular for $m$ and $S$)

$$\widehat{X}_t := X_t^\tau - \int_0^{t\wedge\tau} \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{Z_s} + \int_{t\wedge\tau}^t \frac{d\langle X, m\rangle_s^{\mathbb{F}}}{1 - Z_s}$$

is a $\mathbb{G}$ local martingale. Looking for a deflator of the form $dL_t = L_t \kappa_t d\widehat{m}_t$, and using integration by parts formula, we obtain that, for $\kappa = -(1 - Z)^{-1}$, the process $L(S - S^\tau)$ is a local martingale.

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**Thank you for your attention**