Monitoring in a risk-management framework: inference from mortality experience in a life annuity portfolio

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UNSW-CSIRO WORKSHOP

Risk: Modelling, Optimization and Inference

(with Applications in Finance, Insurance and Superannuation)

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Agenda

- Motivation
- The Enterprise Risk Management framework
- Monitoring. Experience-based assessments
- Stochastic mortality: the basics
- Experience-based adjustment of mortality assumptions (1)
- Experience-based adjustment of mortality assumptions (2)
- Concluding remarks

Presentation mostly based on research and teaching material,
jointly with Annamaria Olivieri (University of Parma)

MOTIVATION

- Longevity risk: one of the critical issues in managing life annuity portfolios and pension funds
- Need for monitoring mortality trends
  - ▷ in populations
  - ▷ in specific portfolios and pension funds
- Analysis and choice of risk management actions (reinsurance, longevity swaps, capital allocation, etc.) should be calibrated on updated information
- The monitoring step in the risk management process should include information updating (in particular concerning mortality trends), relying on sound inferential models

THE ENTERPRISE RISK MANAGEMENT (ERM) FRAMEWORK

ERM (in the insurance context):

- Not an alternative to actuarial mathematics and technique
- Provides a unifying point of view: problems, methods, techniques ranging from awareness of risks to management tools
- Reinterpretation of “intuitive” issues
- New issues arising from a comprehensive approach
- A rigorous framework for
  - ▷ management practice
  - ▷ teaching insurance technique and actuarial mathematics

The ERM process

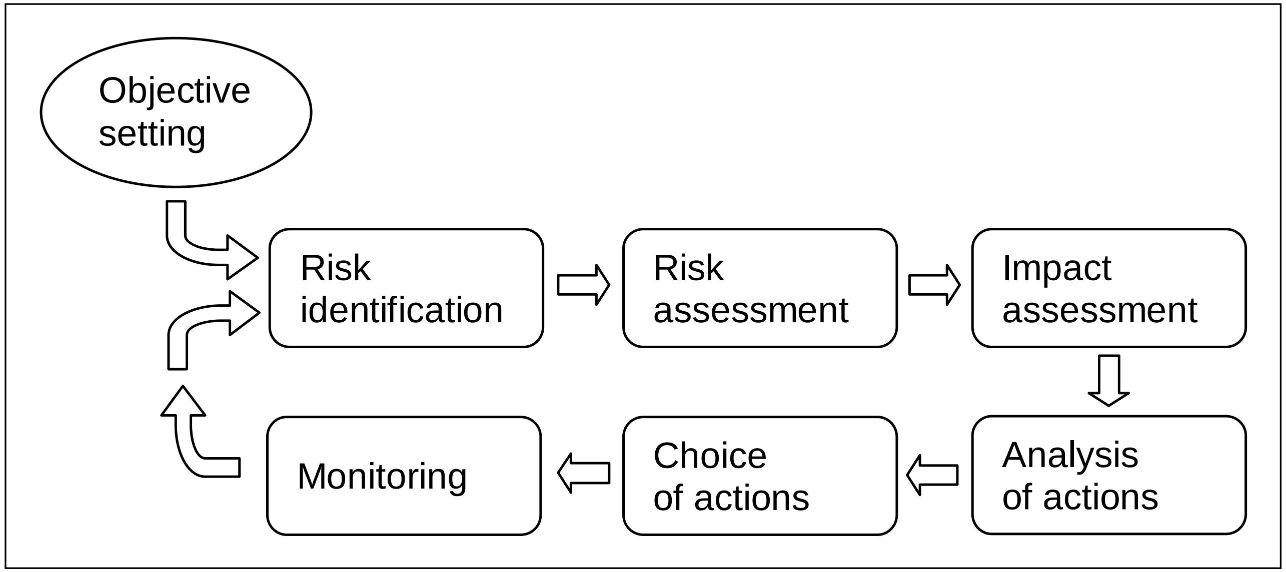


Figure 1 - Steps in the ERM process

We focus on the following steps

- ▷ risk identification
- ▷ risk assessment
- ▷ impact assessment
- ▷ monitoring

Risk assessment, impact assessment, monitoring $\Rightarrow$ *quantitative steps* in the ERM process

Risk identification

See, for example:

International Actuarial Association [2004]

Basic issues

- Risk *sources* (or *causes*) (underwriting, market, operational, etc)
- Risk *components*, in particular:
  - ▷ process risk, i.e. the risk of random fluctuations
  - ▷ systematic risk, i.e. the risk of systematic deviations (in particular because of uncertainty in model choice and/or parameter estimation)
- Risk *factors*, influencing the severity of impact on portfolio results (portfolio size, policy conditions, etc.)

Risk assessment and impact assessment

In general:

- $X_1, X_2, X_3, \ldots$: random variables representing *risks causes*
- $c_1, c_2, c_3, \ldots$: values assigned to decision variables
- Y: a result chosen to express the *impact* of risks

Then:

$$Y = \Phi(X_1, X_2, X_3, \ldots; c_1, c_2, c_3, \ldots)$$

Function Φ should allow (possibly via parameters) for *risk factors*

See the following Figure

The Enterprise Risk Management framework *(cont'd)*

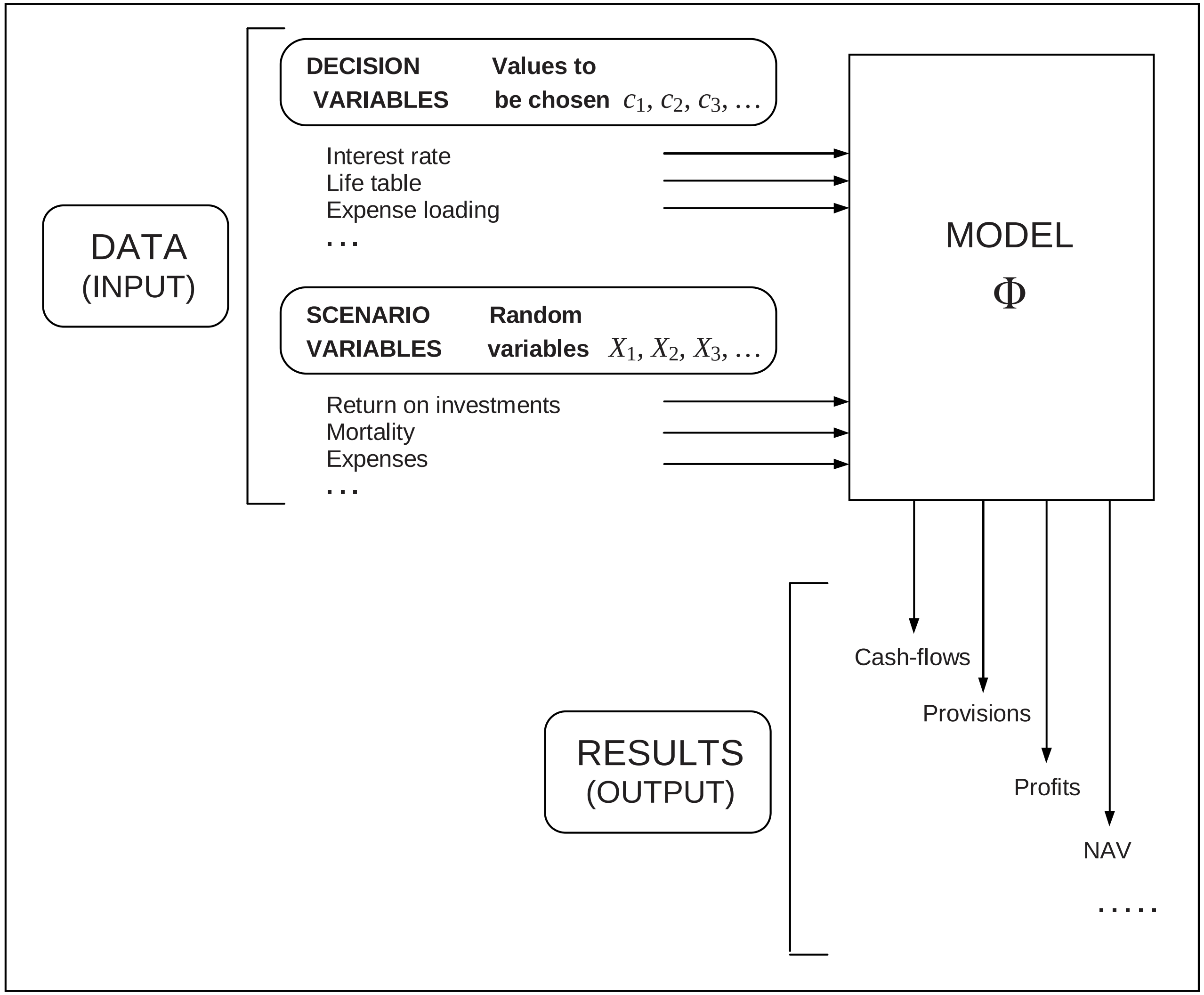


Figure 2 - Modelling for life insurance and life annuities: a comprehensive approach

How to implement the model ?

Ideal target: given

- ▷ the joint prob. distribution of $(X_1, X_2, X_3, \dots)$

or

- ▷ the marginal prob. distributions of $X_1, X_2, X_3, \dots$ and correlation assumptions (possibly via copula)

find the probability distribution of Y

In practice, (almost) impossible to find the probability distribution of Y via analytical procedures (heavy simplifications usually required)

A wide range of approaches available: from purely deterministic to "completely" stochastic

See the following Figures

The Enterprise Risk Management framework *(cont'd)*

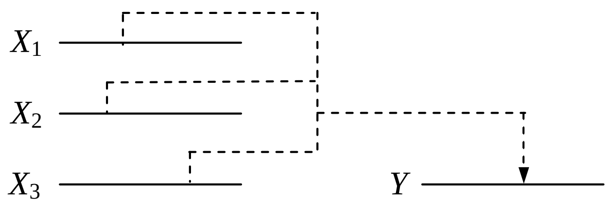
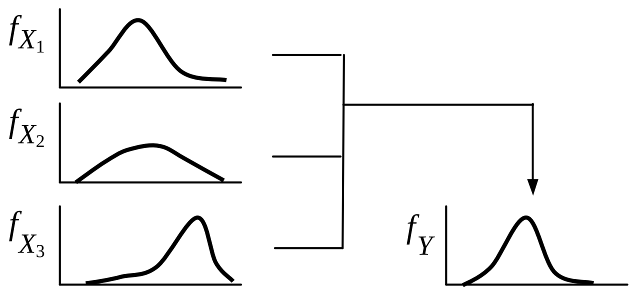
	INPUT OUTPUT	IMPLEMENTATIONS	EXAMPLES
1		**a** - single	- traditional actuarial approach e.g. deterministic Embedded Value - stress testing e.g. Solvency 2
		b - iterative	- scenario testing - sensitivity testing
2	 + CORRELATIONS	**a** - analytical	stochastic risk assessment and impact assessment e.g. - pricing - reserving - capital allocation - reinsurance
		b - analytical approx	
		c - numerical	
		d - simulation	

Figure 3 - Implementing a stochastic model (1-2)

The Enterprise Risk Management framework *(cont'd)*

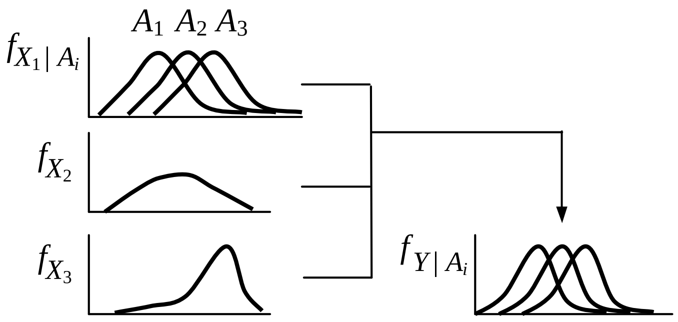
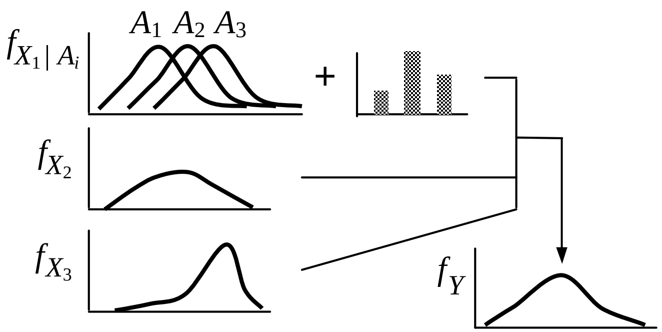
	INPUT / OUTPUT	IMPLEMENTATIONS	EXAMPLES
3	 + CORRELATIONS	**a** - analytical **b** - analytical approx **c** - numerical **d** - simulation	stochastic risk assessment and impact assessment, explicitly allowing for uncertainty risk
4	 + CORRELATIONS	simulation	stochastic risk assessment and impact assessment, explicitly allowing for stochastic assessment of uncertainty risk

Figure 4 - Implementing a stochastic model (3-4)

MONITORING.
EXPERIENCE-BASED ASSESSMENTS

Objectives of the monitoring step (in the ERM process), in particular:

1. checking the effectiveness of the undertaken actions (product design, pricing, reinsurance, swaps, ART, capital allocation, etc.)
2. determining whether changes in the scenario suggest novel solutions
3. adjust probability distributions according to experience
4. . . . . . .

Objective 3 $\Rightarrow$ appropriate modeling structures required, allowing for experience-based adjustments

Monitoring. Experience-based assessments *(cont'd)*

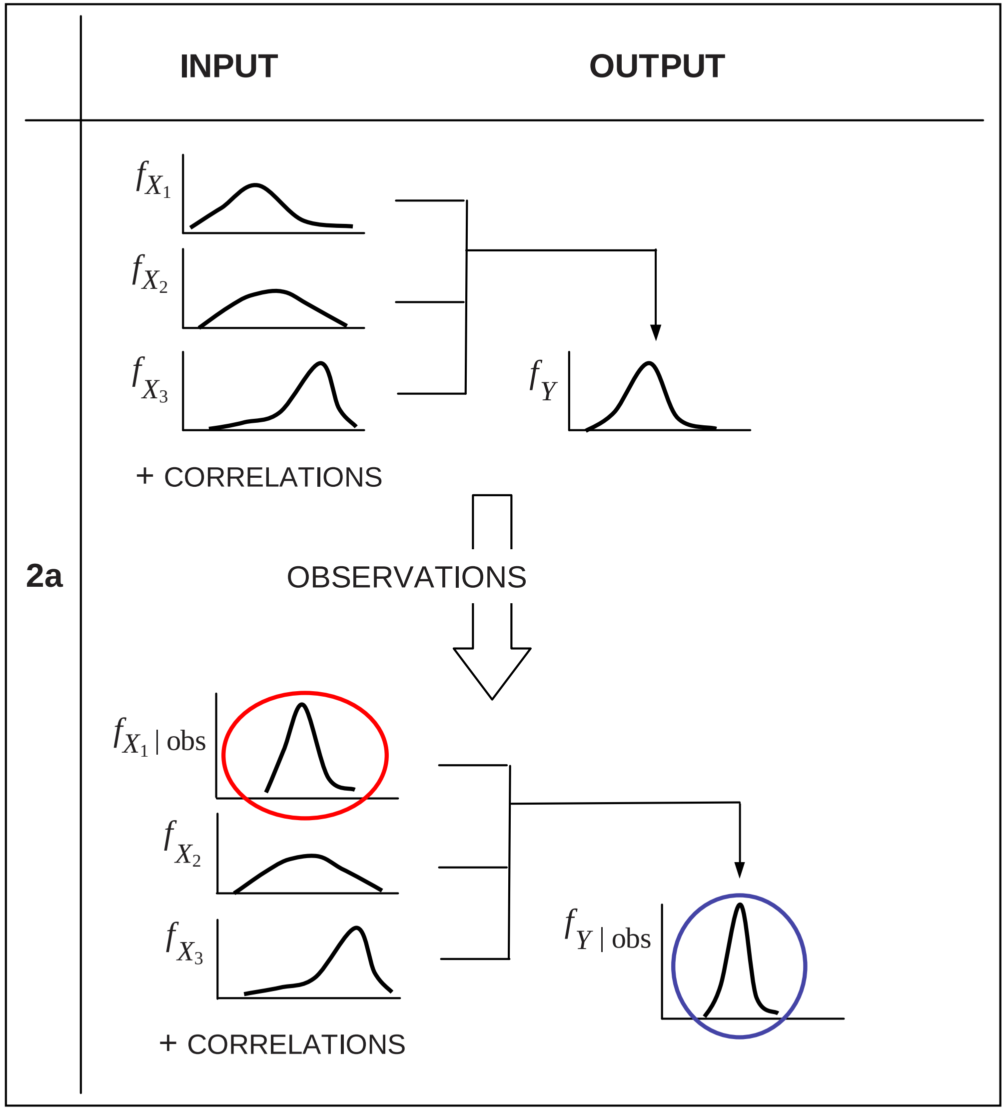


Figure 5 - A stochastic model allowing for experience-based adjustment (2a)

Monitoring. Experience-based assessments *(cont'd)*

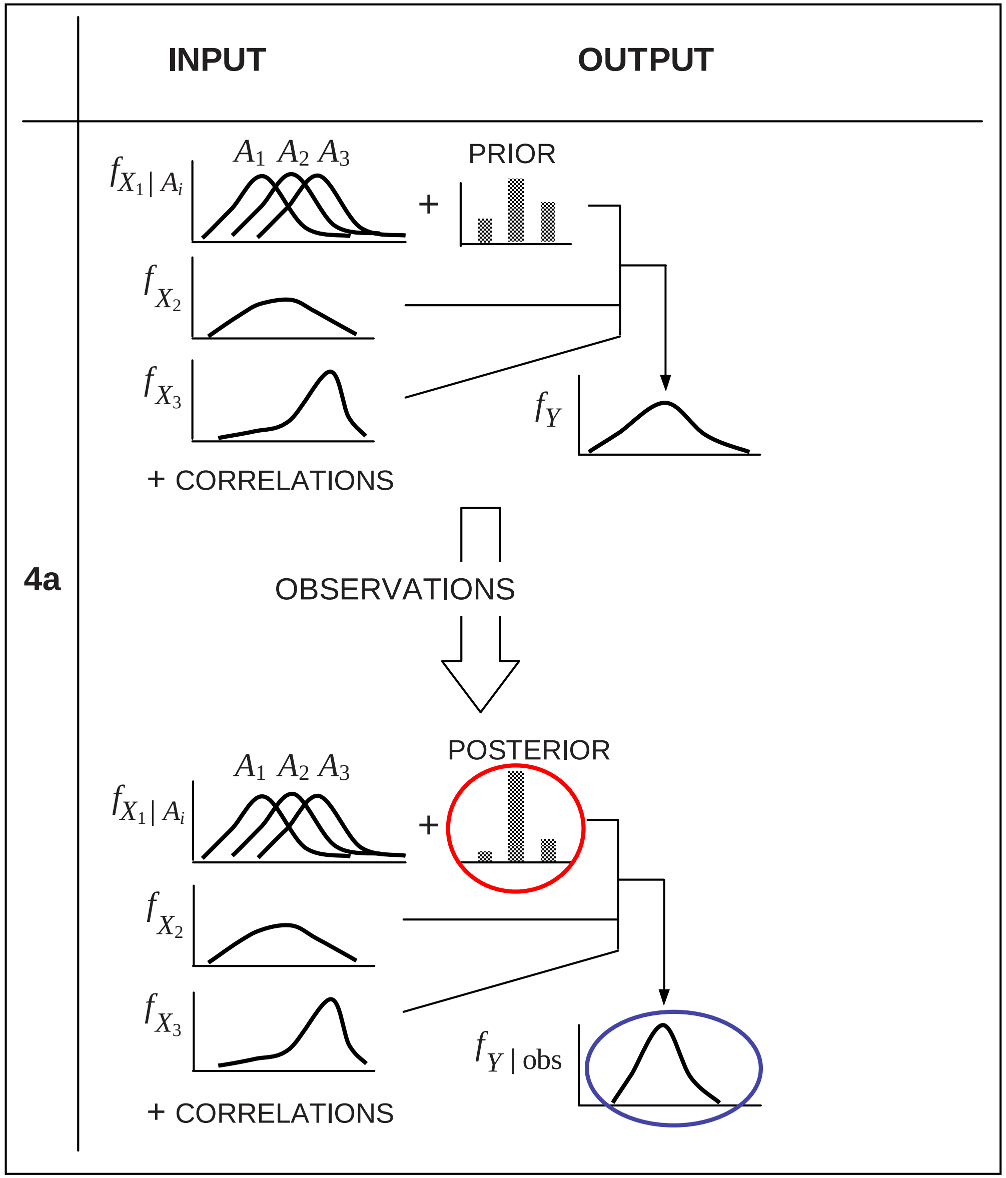


Figure 6 - A stochastic model allowing for experience-based adjustment (4a)

STOCHASTIC MORTALITY: THE BASICS

Risks in life insurance, annuities and pensions

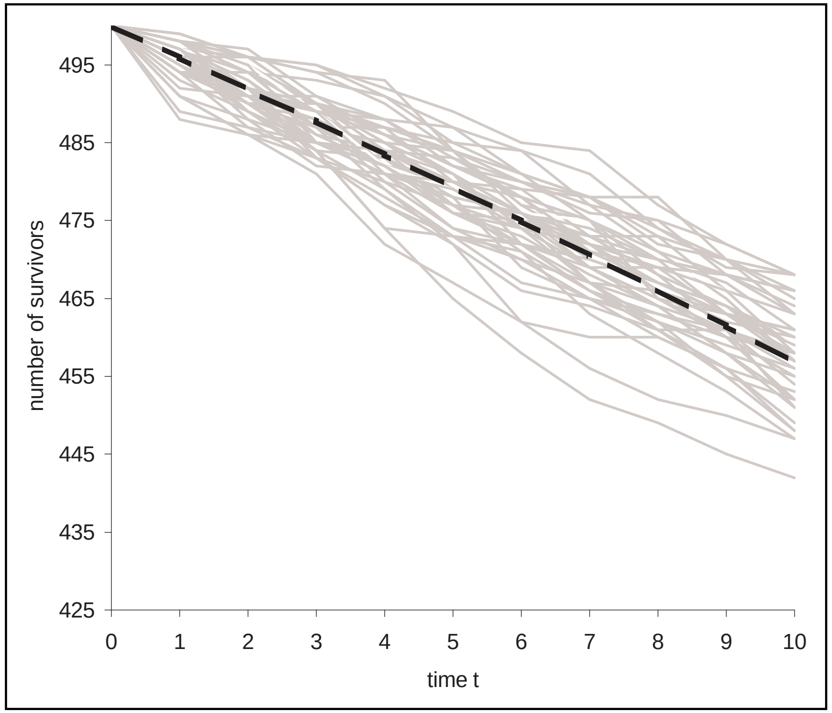
Life insurers and annuity providers take, according to policy conditions (options and guarantees)

- financial risks
- biometric risks (mortality / longevity and disability risks)

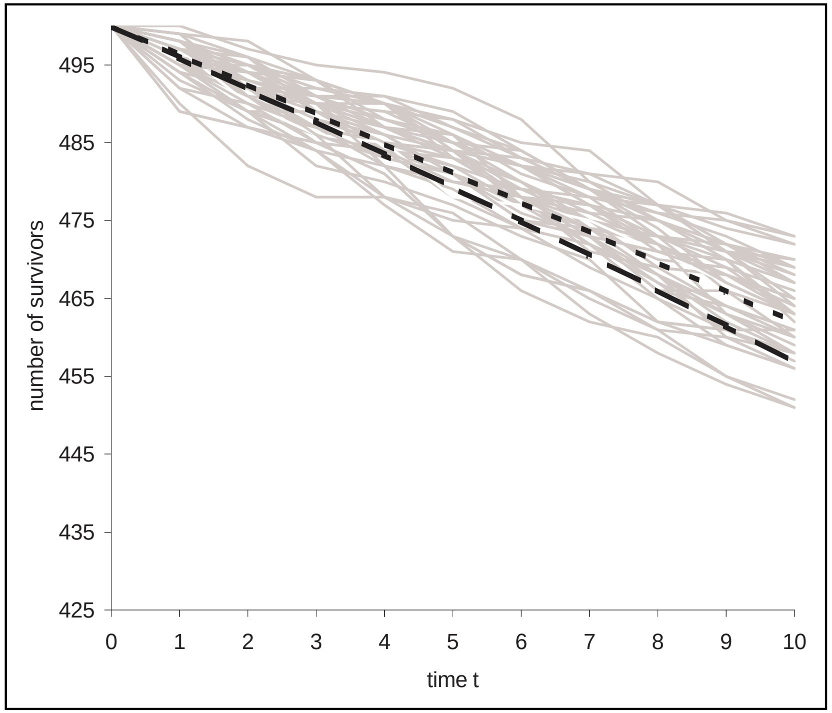
Focus on mortality / longevity risks

Risk arising from *individual lifetimes* is a “process risk” (originated by random mortality fluctuations), called *individual mortality / longevity risk*, and can be diversified by increasing the portfolio size or via reinsurance arrangements, i.e. inside the traditional insurance-reinsurance process

Risk arising from *average lifetime in the portfolio* (originated by systematic deviations, in particular because of future unknown mortality trend) is a “systematic risk”, called *aggregate mortality / longevity risk*, and cannot be diversified inside the traditional insurance-reinsurance process



(a)



(b)

Figure 7 - Simulated number of survivors

(a) random fluctuations $\Rightarrow$ *individual longevity risk*

(b) random fluctuations + systematic deviations $\Rightarrow$ *aggregate longevity risk*

A basic model

Refer to a cohort initially consisting of n_{x_0} individuals age x_0. Define:

$T_{x_0}^{(j)}$ = random lifetime of individual j ($j = 1, 2, \dots, n_{x_0}$)

$$N_{x_0+t} = \sum_{j=1}^{n_{x_0}} I_{\{T_{x_0}^{(j)} > t\}}$$

$$D_{x_0+t} = N_{x_0+t} - N_{x_0+t+1}$$

Assume random lifetimes $T_{x_0}^{(j)}$, $j = 1, 2, \dots, n_{x_0}$, are i.i.d., with probability distribution provided by the life table $\{\ell_x\}_{x=0,1,\dots,\omega}$

Then:

$$N_{x_0+t} \sim \mathrm{Bin}(n_{x_0}, {}_tp_{x_0})$$

and for $z > t$,

$$[N_{x_0+z} \,|\, n_{x_0+t}] \sim \mathrm{Bin}(n_{x_0+t}, {}_{z-t}p_{x_0+t})$$

As regards the numbers D_{x_0+t}, conditional on $N_{x_0+t} = n_{x_0+t}$:

$$[D_{x_0+t} \,|\, n_{x_0+t}] \sim \mathrm{Bin}(n_{x_0+t}, q_{x_0+t})$$

Poisson distribution often adopted as an approximation to the binomial distribution:

$$[D_{x_0+t} \,|\, n_{x_0+t}] \sim \mathrm{Pois}(n_{x_0+t}\, q_{x_0+t})$$

with

$$\mathbb{E}[D_{x_0+t} \,|\, n_{x_0+t}] = n_{x_0+t}\, q_{x_0+t}$$

(under both the Binomial and the Poisson assumption)

Example

$x_0 = 65$; $q_x = \frac{G\,H^x}{1+G\,H^x}$ with $G = 2.005 \times 10^{-6}$, $H = 1.130$

See Figures

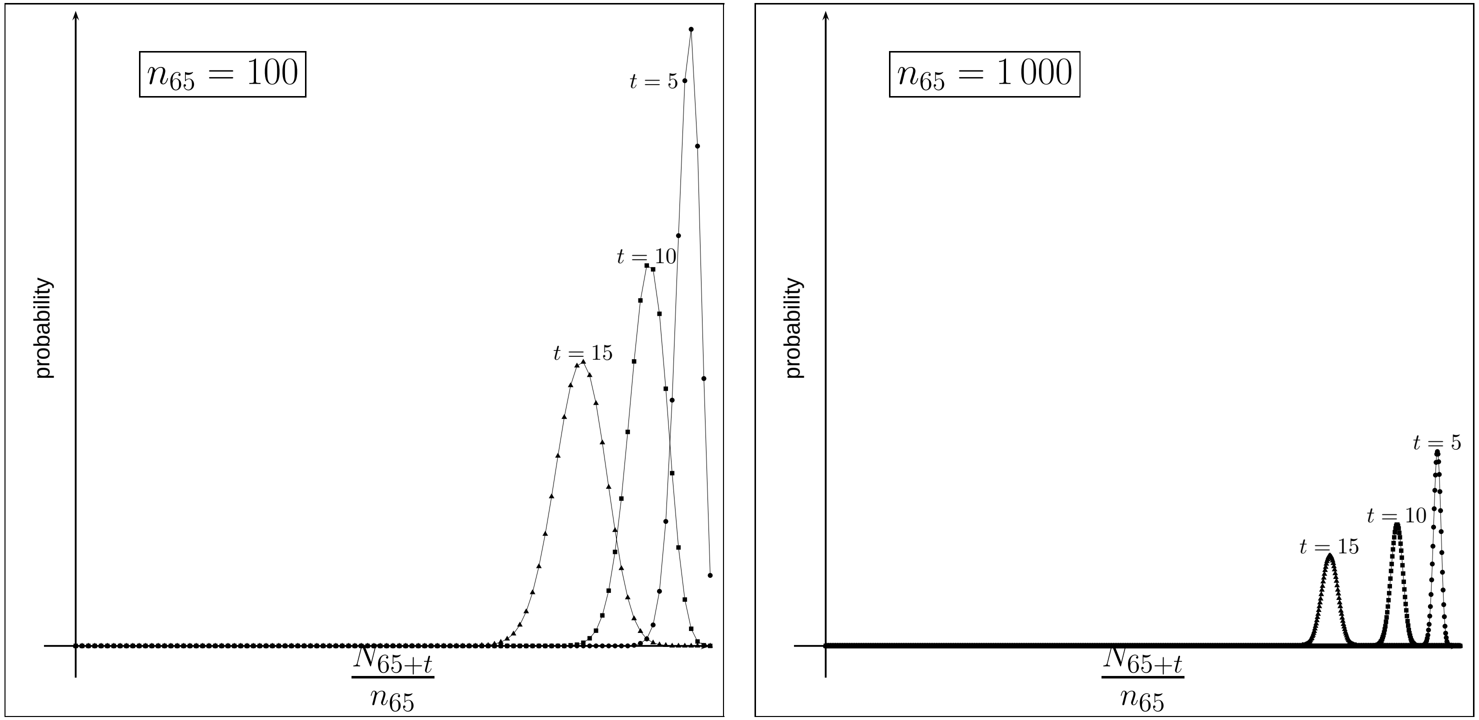


Figure 8 - Probability distribution of $\frac{N_{65+t}}{n_{65}}$; $t = 5, 10, 15$

Example of insurance application

Focus on:

$Y_0^{[\mathrm{P}]}$ = random present value at time 0 of the benefits which will be paid by a portfolio of life annuities (individual annual amount b)

$$Y_0^{[\mathrm{P}]} = b \sum_{t=1}^{\omega - x_0} N_{x_0+t}\,(1+i)^{-t}$$

or, equivalently:

$$Y_0^{[\mathrm{P}]} = b \sum_{j=1}^{n_{x_0}} a_{\,\overline{K_{x_0}^{(j)}}\rceil}$$

We have:

$$\mathbb{E}\left[Y_0^{[\mathrm{P}]}\right] = b \sum_{t=1}^{\omega - x_0} \mathbb{E}[N_{x_0+t}]\,(1+i)^{-t} = b \sum_{t=1}^{\omega - x_0} n_{x_0}\,\frac{\ell_{x_0+t}}{\ell_{x_0}}\,(1+i)^{-t} = n_{x_0}\,b\,a_{x_0}$$

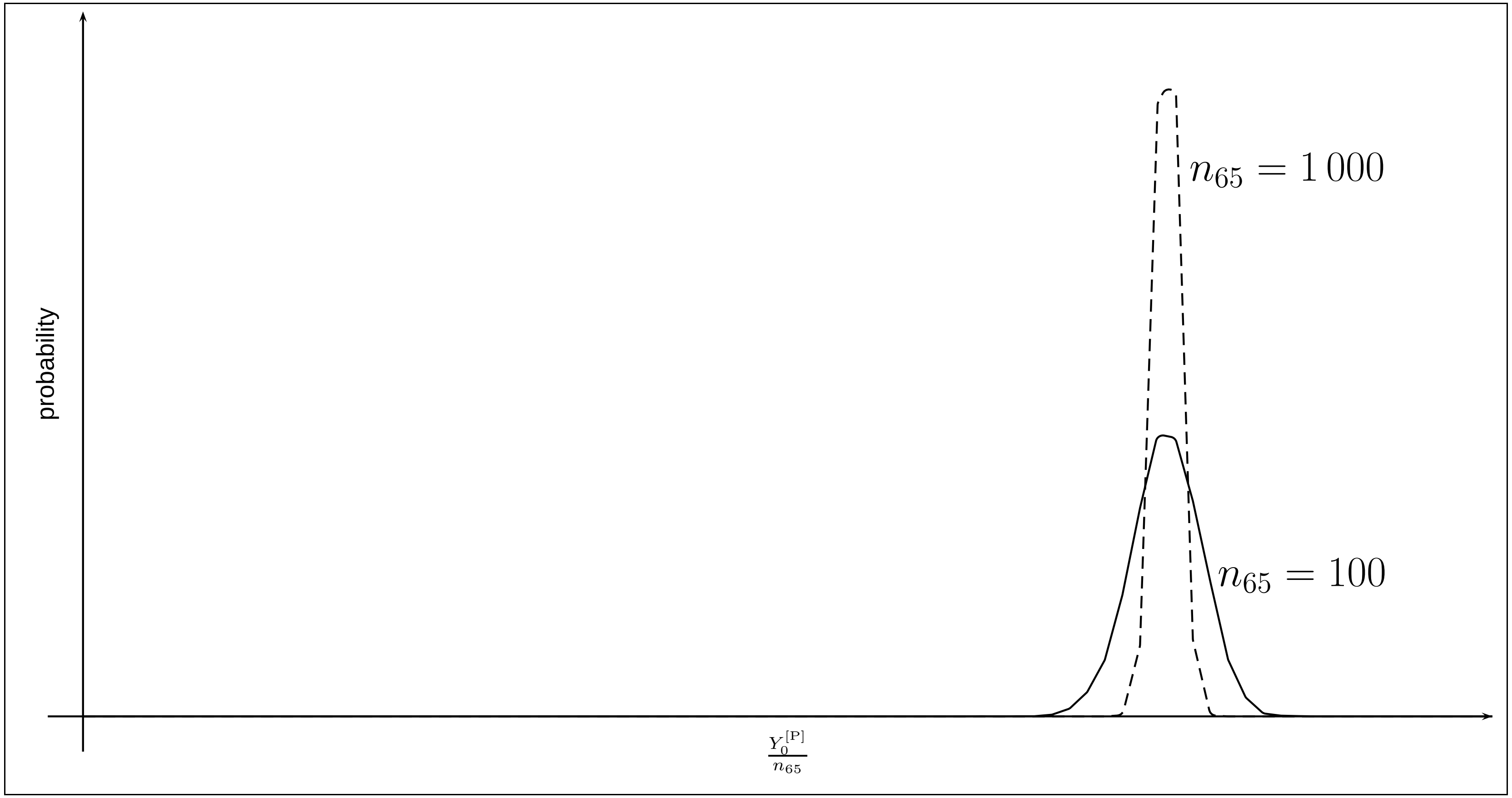


Figure 9 - Probability distribution of $\frac{Y_0^{[P]}}{n_{65}}$ *(*$n_{65} = 100$*;* $n_{65} = 1\,000$*)*

STOCHASTIC MORTALITY: EXPERIENCE-BASED ADJUSTMENTS (1)

A simple approach

The approach basically consists of two steps

1. Choose a set of, say, r scenarios, in order to express alternative hypotheses about future mortality trend:

$$\mathcal{H} = \{H_1, H_2, \dots, H_r\}$$

   - Each scenario: a projected life table (or a survival function, or a force of mortality, etc.)
   - Actuarial applications: scenario testing, assessing the range of variation of quantities such as cash flows, profits, portfolio reserves, etc. $\Rightarrow$ sensitivity analysis

Stochastic mortality: experience-based adjustments (1) *(cont'd)*

2. Assign non-negative normalized weights to the mortality scenarios $\Rightarrow$ a probability distribution on the space $\mathcal{H}$:

$$\rho_1, \ \rho_2, \ \dots \ , \ \rho_r$$

   - Actuarial application: a stochastic approach can be adopted $\Rightarrow$ unconditional (i.e. non conditional on a particular scenario) variances, percentiles, etc., of the value of future cash flows, profits, etc.

Example

See the following Figure

Stochastic mortality: experience-based adjustments (1) *(cont'd)*

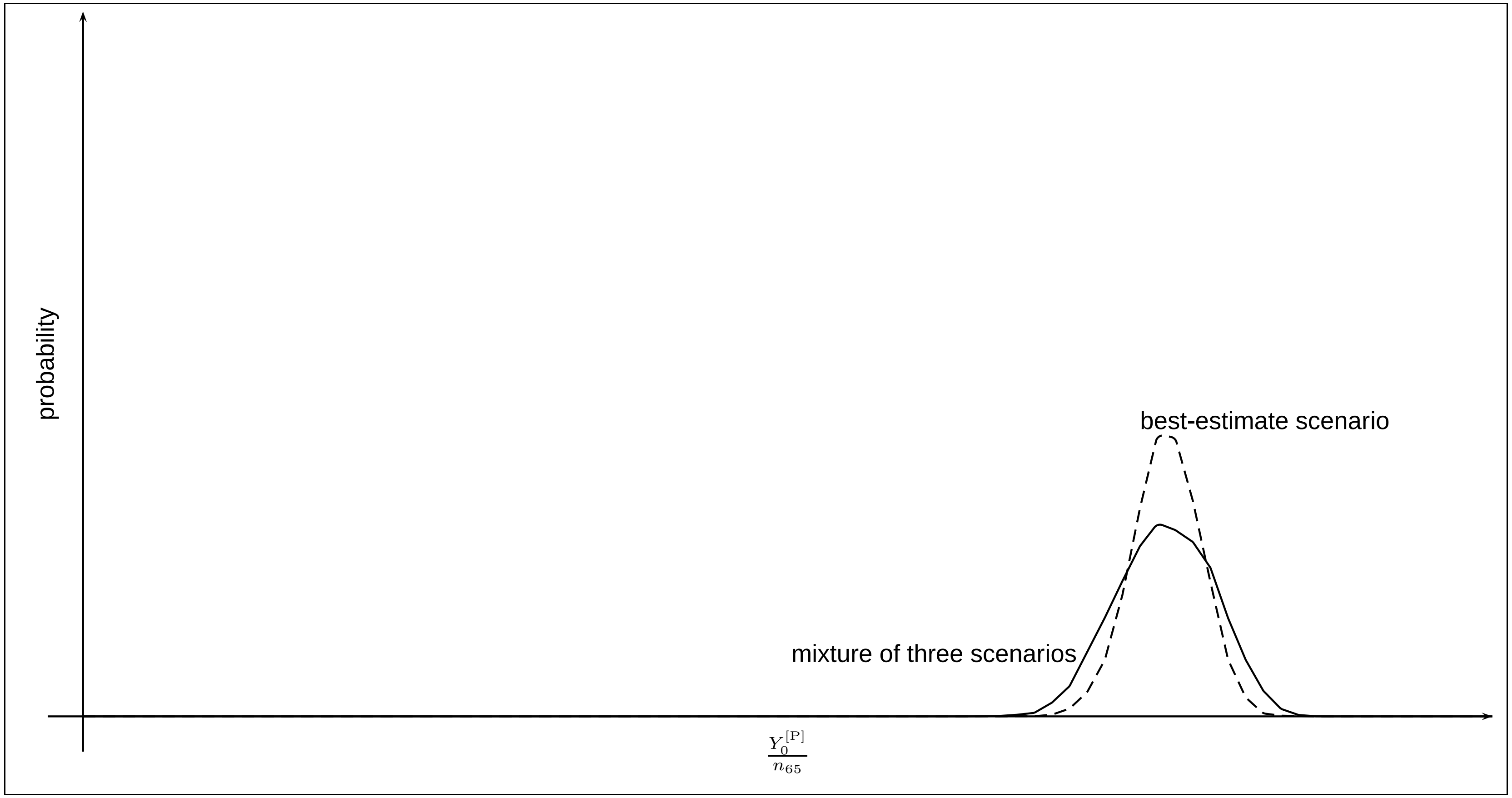


Figure 10 - Conditional and unconditional probability distribution of $\frac{Y_0^{[\mathrm{P}]}}{n_{65}}$ *(*$n_{65} = 100$*)*

Stochastic mortality: experience-based adjustments (1) *(cont'd)*

Main feature: a "static" approach

- ▷ Uncertainty is expressed just in terms of a set $\mathcal{H}$ of assumptions, and the relevant probability distribution ($\Rightarrow$ which one of the assumptions is the best one for describing the aggregate mortality in the cohort)
- ▷ No future shift from such a trend is allowed for in the stochastic model
- ▷ Critical aspect: assumptions about the temporal correlation of changes in the probabilities of death are implicitly involved
- ▷ Possible mortality shocks are not embedded into the static representation (not a problem when dealing with life annuities)
- ▷ Updates of the weights ρ 's based on experience could be introduced, while keeping the setting as a static one

Modeling with a discrete set of scenarios

See Olivieri and Pitacco [2002]

Notation

- $f(t, y)$: pdf of individual lifetime T, referred to individuals born in year y
- $H(y)$: hypothesis about mortality trend for people born in year y
- family of pdf's:

$$\{f(t, y \,|\, H(y)); H(y) \in \mathcal{H}(y)\}$$

- in particular:

$$\{f(t, y \,|\, \theta(y));\ \theta(y) \in \Theta(y)\}$$

  where:
  - ▷ $\theta(y)$ = vector-valued (in particular, real-valued) parameter
  - ▷ $\Theta(y)$ = the parameter space

Stochastic mortality: experience-based adjustments (1) *(cont'd)*

We focus on the parameterized setting, and refer to a parameter finite space

For simplicity, we address one generation, hence:

$$\{f(t \,|\, \theta); \theta \in \Theta\}$$

For the random parameter $\tilde{\theta}$, prior distribution:

$$g(\theta) = \Pr[\tilde{\theta} = \theta]$$

Conditional and unconditional expected values can be calculated

In particular, the following result holds:

$$\underbrace{\mathbb{V}\mathrm{ar}[T]}_{\text{unconditional variance}} = \underbrace{\mathbb{E}\big[\mathbb{V}\mathrm{ar}[T \,|\, \tilde{\theta}]\big]}_{\text{random fluctuations}} + \underbrace{\mathbb{V}\mathrm{ar}\big[\mathbb{E}[T \,|\, \tilde{\theta}]\big]}_{\text{systematic deviations}}$$

Inference: the model

Refer to a homogeneous set of n individuals (same generation), considered at time (=age) τ

Let

- $T_h - \tau$ = remaining lifetime of individual h
- $T_1, T_2, \dots, T_n$ iid conditional on any given scenario

Sampling pdf

$$f_\tau(t \,|\, \theta) = \begin{cases} 0 & \text{if } t \leq \tau \\ \dfrac{f(t \,|\, \theta)}{\int_\tau^{+\infty} f(u \,|\, \theta)\, \mathrm{d}u} & \text{if } t > \tau \end{cases}$$

Multivariate sampling pdf

$$f_\tau(t_1, t_2, \dots, t_n \,|\, \theta) = \prod_{h=1}^{n} f_\tau(t_h \,|\, \theta)$$

Stochastic mortality: experience-based adjustments (1) *(cont'd)*

(Prior) predictive pdf (restricted to $[\tau, +\infty)$)

$$f_\tau(t) = \sum_{\theta \in \Theta} f_\tau(t \,|\, \theta)\, g(\theta)$$

Steps of the inferential procedure

- ▷ Observation: m deaths in the age interval $[\tau, \tau']$, at ages $\underline{x} = (x_1, x_2, \ldots, x_m)$
- ▷ Update the opinion about the possible mortality trend (i.e. about the probability distribution over Θ) $\Rightarrow$ posterior:

$$g(\theta \,|\, m, \underline{x}) \propto g(\theta)\, L(\theta \,|\, m, \underline{x})$$

- ▷ Calculate the (posterior) predictive pdf:

$$f_\tau(t \,|\, m, \underline{x}) = \sum_{\theta \in \Theta} f_\tau(t \,|\, \theta)\, g(\theta \,|\, m, \underline{x})$$

Inference: implementation and numerical examples

Weibull law:

$$f(t \,|\, \alpha, \beta) = \frac{\alpha}{\beta} \left(\frac{t}{\beta} \right)^{\alpha - 1} \exp \left(- \left(\frac{t}{\beta} \right)^{\alpha} \right)$$

Finite parameter space:

$$\Theta = \left\{ (\alpha_i, \beta_j) \right\}$$

$$g(\alpha_i, \beta_j) = \mathbb{P}[\tilde{\alpha} = \alpha_j \wedge \tilde{\beta} = \beta_j]$$

Numerical implementation:

$$\Theta = \left\{ (\alpha_i, \beta_j);\; i = 1, \ldots, 5,\; j = 1, \ldots, 5 \right\}$$

See following Tables

Stochastic mortality: experience-based adjustments (1) *(cont'd)*

α	β				
	82	83.5	85.2	87	89
7.00	16.097	17.187	18.450	19.816	21.364
8.00	15.548	16.680	17.991	19.411	21.021
9.15	15.155	16.331	17.695	19.170	20.841
10.45	14.902	16.126	17.542	19.072	20.802
12.00	14.764	16.036	17.506	19.090	20.877

Space Θ: $\mathbb{E}[T - 65 \,|\, T > 65; \alpha, \beta]$

α	β				
	82	83.5	85.2	87	89
7.00	82.599	90.181	99.035	108.646	119.473
8.00	69.555	76.119	83.758	92.042	101.422
9.15	58.894	64.518	71.013	77.999	85.857
10.45	50.135	54.895	60.337	66.126	72.563
12.00	42.406	46.326	50.749	55.389	60.477

Space Θ: $\mathbb{V}\mathrm{ar}[T - 65 \,|\, T > 65; \alpha, \beta]$

Stochastic mortality: experience-based adjustments (1) *(cont'd)*

α	β				
	82	83.5	85.2	87	89
7.00	0.0025	0.0075	0.03	0.0075	0.0025
8.00	0.0075	0.0225	0.09	0.0225	0.0075
9.15	0.0300	0.0900	0.36	0.0900	0.0300
10.45	0.0075	0.0225	0.09	0.0225	0.0075
12.00	0.0025	0.0075	0.03	0.0075	0.0025

Prior: $g^{(1)}(\alpha, \beta)$

α	β				
	82	83.5	85.2	87	89
7.00	0.04	0.04	0.04	0.04	0.04
8.00	0.04	0.04	0.04	0.04	0.04
9.15	0.04	0.04	0.04	0.04	0.04
10.45	0.04	0.04	0.04	0.04	0.04
12.00	0.04	0.04	0.04	0.04	0.04

Prior: $g^{(2)}(\alpha, \beta)$

Posterior g calculated in several *cases* $\Rightarrow$ posterior predictive f
$\Rightarrow$ markers of the remaining lifetime

Each case defined by the triplet:

(prior g, actual scenario, simulation)

where

- prior: $g^{(1)}$ or $g^{(2)}$
- actual scenario (i.e. scenario assumed in the simulation procedure):

$$S_h = (\alpha_h, \beta_h); \quad h = 1, 2, \dots, 5$$

  Note: according to $g^{(1)}$, $S_3 = (\alpha_3, \beta_3)$ is the "best estimate" scenario
- simulation $\Rightarrow$ number of deaths in the interval $[\tau, \tau'] = [60, 65]$
  - ▷ SD = systematic deviations
  - ▷ RF = random fluctuations (only)

Stochastic mortality: experience-based adjustments (1) *(cont'd)*

	Prior	Posterior				
		S_1	S_2	S_3	S_4	S_5
$\mathbb{E}[T-65 \,\vert\, T>65]$	17.792	16.677	17.092	17.675	18.417	19.352
$\mathbb{E}\big[\mathbb{V}\mathrm{ar}[T-65 \,\vert\, T>65; \tilde{\alpha}, \tilde{\beta}]\big]$	71.921	82.952	77.779	70.998	65.567	58.739
$\mathbb{V}\mathrm{ar}\big[\mathbb{E}[T-65 \,\vert\, T>65; \tilde{\alpha}, \tilde{\beta}]\big]$	1.466	1.030	1.605	0.815	1.807	1.696
$\mathbb{V}\mathrm{ar}[T-65 \,\vert\, T>65]$	73.387	83.982	79.384	71.813	67.374	60.435
$\mathrm{Mode}[T]$	84.072	81.298	82.429	83.940	84.941	86.368

Cases ($g^{(1)}$, S_h, SD): prior and posterior markers of the remaining lifetime

	Prior	Posterior				
		S_1	S_2	S_3	S_4	S_5
$\mathbb{E}[T-65 \,\vert\, T>65]$	17.979	16.351	17.097	17.454	18.323	20.159
$\mathbb{E}\big[\mathbb{V}\mathrm{ar}[T-65 \,\vert\, T>65; \tilde{\alpha}, \tilde{\beta}]\big]$	73.621	82.218	81.003	70.609	62.427	60.075
$\mathbb{V}\mathrm{ar}\big[\mathbb{E}[T-65 \,\vert\, T>65; \tilde{\alpha}, \tilde{\beta}]\big]$	4.189	0.502	2.502	4.752	3.362	1.045
$\mathbb{V}\mathrm{ar}[T-65 \,\vert\, T>65]$	77.810	82.720	83.505	75.361	65.789	61.120
$\mathrm{Mode}[T]$	83.923	80.783	82.039	82.804	84.586	87.416

Cases ($g^{(2)}$, S_h, SD): prior and posterior markers of the remaining lifetime

Stochastic mortality: experience-based adjustments (1) *(cont'd)*

	Prior	Posterior				
		S_1	S_2	S_3	S_4	S_5
$\mathbb{E}[T-65 \mid T>65]$	17.792	17.764	17.750	17.725	17.710	17.645
$\mathbb{E}\big[\mathbb{V}\mathrm{ar}[T-65 \mid T>65; \tilde{\alpha}, \tilde{\beta}]\big]$	71.921	72.416	72.191	71.801	71.541	70.549
$\mathbb{V}\mathrm{ar}\big[\mathbb{E}[T-65 \mid T>65; \tilde{\alpha}, \tilde{\beta}]\big]$	1.466	0.829	0.824	0.816	0.814	0.823
$\mathbb{V}\mathrm{ar}[T-65 \mid T>65]$	73.388	73.245	73.015	72.618	72.355	71.371
$\mathrm{Mode}[T]$	84.072	84.009	83.999	83.981	83.969	83.913

Cases ($g^{(1)}$, S_h, RF): prior and posterior markers of the remaining lifetime

STOCHASTIC MORTALITY: EXPERIENCE-BASED ADJUSTMENTS (2)

Focus on longevity risk (both individual and aggregate)

See: Olivieri and Pitacco [2009], Olivieri and Pitacco [2012]

Preliminary ideas

Refer to a portfolio of life annuities (one cohort or multi-cohort)

Assume that:

- a life table, providing a best-estimate of annuitants' mortality, is available to the insurer
- the insurer has no access to the data sets and the methodology underlying the construction of the life table
- awareness of uncertainty in future mortality trends $\Rightarrow$ the life table is used as the basic input of a stochastic mortality model

The model

We generalize to multi-cohort cases the basic model already described

Notation:

- t_0 = starting time of the life annuity portfolio
- x_0 = annuitants' age at entry
- t = portfolio past duration since time t_0; $t = 0, 1, 2, \dots$
- $D_{x,t}$ = random number of deaths in year $(t-1, t)$ for those aged x at time $t-1$
- $N_{x,t}$ = random number of individuals alive age x at time t
- $d_{x,t}$, $n_{x,t}$ = possible (and observed) outcomes of the random variables $D_{x,t}$, $N_{x,t}$ respectively

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

Probability distribution (under usual assumptions):

$$[D_{x,t} \,|\, q_{x,t}; n_{x,t-1}] \sim \mathrm{Bin}(n_{x,t-1}, q_{x,t})$$

with $q_{x,t}$ = assumed probability of death (possibly the best-estimate)

Approximation:

$$[D_{x,t} \,|\, q_{x,t}; n_{x,t-1}] \sim \mathrm{Pois}(n_{x,t-1}\, q_{x,t})$$

Uncertainty about the mortality trend $\Rightarrow$ $Q_{x,t}$ = random mortality rate

Modelling approaches:

1. assign a probability distribution to $Q_{x,t}$
2. let

$$Q_{x,t} = q^*_{x,t}\, Z_{x,t}$$

   with $Z_{x,t}$ = (positive) random adjustment to the best-estimate mortality rate $q^*_{x,t}$; assign a probability distribution to $Z_{x,t}$

Modelling approach 1

Assume:

$$Q_{x,t} \sim \mathrm{Beta}(a_{x,t}, b_{x,t})$$

Unconditional distribution of the number of deaths then follows the Beta-Binomial law:

$$\mathbb{P}[D_{x,t} = d \,|\, n_{x,t-1}] = \binom{n_{x,t-1}}{d} \frac{\Gamma(a_{x,t} + b_{x,t})}{\Gamma(a_{x,t})\,\Gamma(b_{x,t})} \frac{\Gamma(a_{x,t} + d)\,\Gamma(b_{x,t} + n_{x,t-1} - d)}{\Gamma(a_{x,t} + b_{x,t} + n_{x,t-1})}$$

where $\Gamma(\cdot)$ is the incomplete Gamma function

We have:

$$\mathbb{E}[Q_{x,t}] = \frac{a_{x,t}}{a_{x,t} + b_{x,t}}$$

$$\mathbb{E}[D_{x,t} \,|\, n_{x,t-1}] = n_{x,t-1} \frac{a_{x,t}}{a_{x,t} + b_{x,t}}$$

Note that:

$$\mathbb{E}[D_{x,t} \,|\, q^*_{x,t}; n_{x,t-1}] = n_{x,t-1}\; q^*_{x,t}$$

Then

$$\mathbb{E}[D_{x,t} \,|\, n_{x,t-1}] \gtreqless \mathbb{E}[D_{x,t} \,|\, q^*_{x,t}; n_{x,t-1}]$$

depending on the comparison between $q^*_{x,t}$ and $\mathbb{E}[Q_{x,t}]$

Modelling approach 2

Assume:

$$Z_{x,t} \sim \mathrm{Gamma}(\alpha_{x,t}, \beta_{x,t})$$

It turns out:

$$Q_{x,t} \sim \mathrm{Gamma}\left(\alpha_{x,t}, \frac{\beta_{x,t}}{q^*_{x,t}}\right)$$

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

and

$$[D_{x,t} | \, n_{x,t-1}] \sim \mathrm{NegBin}\left(\alpha_{x,t}, \frac{\theta_{x,t}}{\theta_{x,t}+1}\right) \qquad (*)$$

with

$$\theta_{x,t} = \frac{\beta_{x,t}}{n_{x,t-1}\, q^*_{x,t}}$$

Result $(*)$ generalizes the well known Poisson-Gamma structure

We have:

$$\mathbb{E}[Q_{x,t}] = \frac{\alpha_{x,t}}{\beta_{x,t}}\, q^*_{x,t}$$

$$\mathbb{E}[D_{x,t} | n_{x,t-1}] = \frac{\alpha_{x,t}}{\theta_{x,t}} = \frac{\alpha_{x,t}}{\beta_{x,t}}\, n_{x,t-1}\, q^*_{x,t}$$

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

Note that:

$$\mathbb{E}[D_{x,t}|n_{x,t-1}] \gtreqless \mathbb{E}[D_{x,t}|q^*_{x,t}; n_{x,t-1}]$$

depending on the value of $\dfrac{\alpha_{x,t}}{\beta_{x,t}}$ ($\Rightarrow$ systematic deviations in mortality)

Advantages / disadvantages of the approach

- the Gamma distribution does not guarantee that the mortality rate is bounded in $(0, 1)$ (in particular at the oldest ages, when $q^*_{x,t}$ is high, while $n_{x,t-1}$ is presumably low)
- the model leads quite naturally to a dynamic setting, through a Bayesian inferential procedure, allowing
  - ▷ to account for correlations among the $Z_{x,t}$'s
  - ▷ to update parameters to experience
- approximation errors at the older ages may become negligible when a portfolio consisting of multiple cohorts is addressed

Rationale of experience-based assessments: a ratio $\frac{q_{x,t}}{q^*_{x,t}} < 1$ in year $(t-1, t)$ (where $q_{x,t}$ = realized value of $Q_{x,t}$) is quite always followed by a ratio $\frac{q_{x+1,t+1}}{q^*_{x+1,t+1}} < 1$ (and also $\frac{q_{x,t+1}}{q^*_{x,t+1}} < 1$) in the following year

One cohort. The inferential procedure

Refer to

- one cohort (the model can be applied to the multi-cohort case)
- the Poisson-Gamma model (approach 2)

Assume, for all x and all t:

$$Z_{x,t} \sim \mathrm{Gamma}\left(\bar{\alpha}, \bar{\beta}\right)$$

For example, $\bar{\alpha}, \bar{\beta}$ such that $\mathbb{E}[Q_{x,t}] = q^*_{x,t}$ ($\Rightarrow$ one degree of freedom)

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

It follows:

$$[D_{x_0,1}|n_{x_0,0}] \sim \mathrm{NegBin}\left(\bar{\alpha}, \frac{\theta_{x_0,1}}{\theta_{x_0,1}+1}\right)$$

where

$$\theta_{x_0,1} = \frac{\bar{\beta}}{n_{x_0,0}\, q^*_{x_0,1}}$$

Let $d_{x_0,1}$ denote the number of deaths observed in year $(0,1)$

Then,

$$n_{x_0+1,1} = n_{x_0,0} - d_{x_0,1}$$

Posterior probability distribution of $Q_{x_0,1}$ conditional on $D_{x_0,1} = d_{x_0,1}$:

$$[Q_{x_0,1}|d_{x_0,1}] \sim \mathrm{Gamma}\left(\bar{\alpha} + d_{x_0,1}, \frac{\bar{\beta}}{q^*_{x_0,1}} + n_{x_0,0}\right)$$

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

Posterior probability distribution of $Z_{x,t}$ conditional on $D_{x_0,1} = d_{x_0,1}$:

$$[Z_{x,t} | d_{x_0,1}] \sim \mathrm{Gamma}\left(\bar{\alpha} + d_{x_0,1}, \bar{\beta} + n_{x_0,0}\, q^*_{x_0,1}\right) \qquad (^\circ)$$

Expected values of $Z_{x,t}$

- ▷ prior

$$\mathbb{E}[Z_{x,t}] = \frac{\bar{\alpha}}{\bar{\beta}}$$

- ▷ posterior at time 1

$$\mathbb{E}[Z_{x,t} | d_{x_0,1}] = \frac{\bar{\alpha} + d_{x_0,1}}{\bar{\beta} + n_{x_0,0}\, q^*_{x_0,1}}$$

$\Rightarrow$ $\mathbb{E}[Z_{x,t} | d_{x_0,1}] \gtreqless \mathbb{E}[Z_{x,t}]$ depending on the comparison between $d_{x_0,1}$ and the relevant expected value $n_{x_0,0}\, q^*_{x_0,1}$

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

Valuations performed at time 1 involving the next year:

$$[Q_{x_0+1,2} | d_{x_0,1}] \sim \mathrm{Gamma}\left(\bar{\alpha} + d_{x_0,1}, \frac{\bar{\beta} + n_{x_0,0}\, q^*_{x_0,1}}{q^*_{x_0+1,2}}\right)$$

and hence

$$[D_{x_0+1,2} | n_{x_0,0}, d_{x_0,1}] \sim \mathrm{NegBin}\left(\bar{\alpha} + d_{x_0,1}, \frac{\theta_{x_0+1,2}}{\theta_{x_0+1,2} + 1}\right)$$

with

$$\theta_{x_0+1,2} = \frac{\bar{\beta} + n_{x_0,0}\, q^*_{x_0,1}}{n_{x_0+1,1}\, q^*_{x_0+1,2}}$$

Similar steps at times $t = 2, 3, \dots$

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

Focus on the expected number of deaths in each year $\Rightarrow$ at time $t-1$ ($t = 1, 2, \dots$) we have:

$$\mathbb{E}[D_{x_0+t-1,t} \,|\, n_{x_0,0}, d_{x_0,1}, d_{x_0+1,2}, \dots, d_{x_0+t-2,t-1}]$$

$$= \underbrace{\frac{\bar{\alpha} + \sum_{h=1}^{t-1} d_{x_0+h-1,h}}{\bar{\beta} + \sum_{h=1}^{t-1} n_{x_0+h-1,h-1}\, q^*_{x_0+h-1,h}}}_{A} \; \underbrace{n_{x_0+t-1,t-1}\, q^*_{x_0+t-1,t}}_{B}$$

- B = expected value of $D_{x_0+t-1,t}$ conditional on best-estimate $q^*_{x_0+t-1,t}$
- A = adjustment coefficient, updated to the observed number of deaths in respect of those expected at the beginning of each year
  - ▷ experience consistent with what expected $\Rightarrow$ coefficient will remain stable in time
  - ▷ number of deaths lower than expected $\Rightarrow$ coefficient will decrease in time

Numerical findings

Refer to the expected systematic deviation $\mathbb{E}[Z_t|\{d_s\}_{s=1,2,\ldots,t}]$

Choose the best-estimate life table $\{q^*\}$

At time 0 set:

- $\bar{\alpha} = \bar{\beta}$ ($\Leftarrow$ meaning of the best-estimate life table)
- $\bar{\beta} = 100$ ($\Leftarrow$ expert's judgment on volatility; see below for alternative choices)

Then:

$$\mathbb{E}[Q_{x,t}] = q^*_{x,t}$$

$$\mathbb{V}\mathrm{ar}[Q_{x,t}] = \frac{\bar{\alpha}}{(\bar{\beta})^2}\,(q^*_{x,t})^2$$

$$\mathbb{CV}[Q_{x,t}] = \frac{\sqrt{\mathbb{V}\mathrm{ar}[Q_{x,t}]}}{\mathbb{E}[Q_{x,t}]} = \frac{1}{\sqrt{\bar{\alpha}}} = 10\%$$

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

Some results

- ▷ Figure 11: it is assumed $d_{x,s} = n_{x,s-1}\, q^*_{x,s}$ for $s = 1, 2, \dots, t$
- ▷ Figure 12: it is assumed $d_{x,s} = 0.75\, n_{x,s-1}\, q^*_{x,s}$ $\Rightarrow$ adjustments
- ▷ Figure 13: it is assumed $d_{x,s} = 1.25\, n_{x,s-1}\, q^*_{x,s}$ $\Rightarrow$ adjustments
- ▷ Figures 14 - 16: alternative values for $\bar{\beta}$ are considered (joint with $\bar{\alpha} = \bar{\beta}$)
  - $\bar{\beta} = 100 \;\Rightarrow\; \mathbb{CV}[Q_{x,t}] = 0.10$
  - $\bar{\beta} = 25 \;\Rightarrow\; \mathbb{CV}[Q_{x,t}] = 0.20$
  - $\bar{\beta} = 400 \;\Rightarrow\; \mathbb{CV}[Q_{x,t}] = 0.05$

  $\Rightarrow$ effects of the assumed volatility (in terms of $\mathbb{CV}$) of the mortality rate, on the expected systematic deviation

Note: numerical results also refer to the multi-cohort case

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

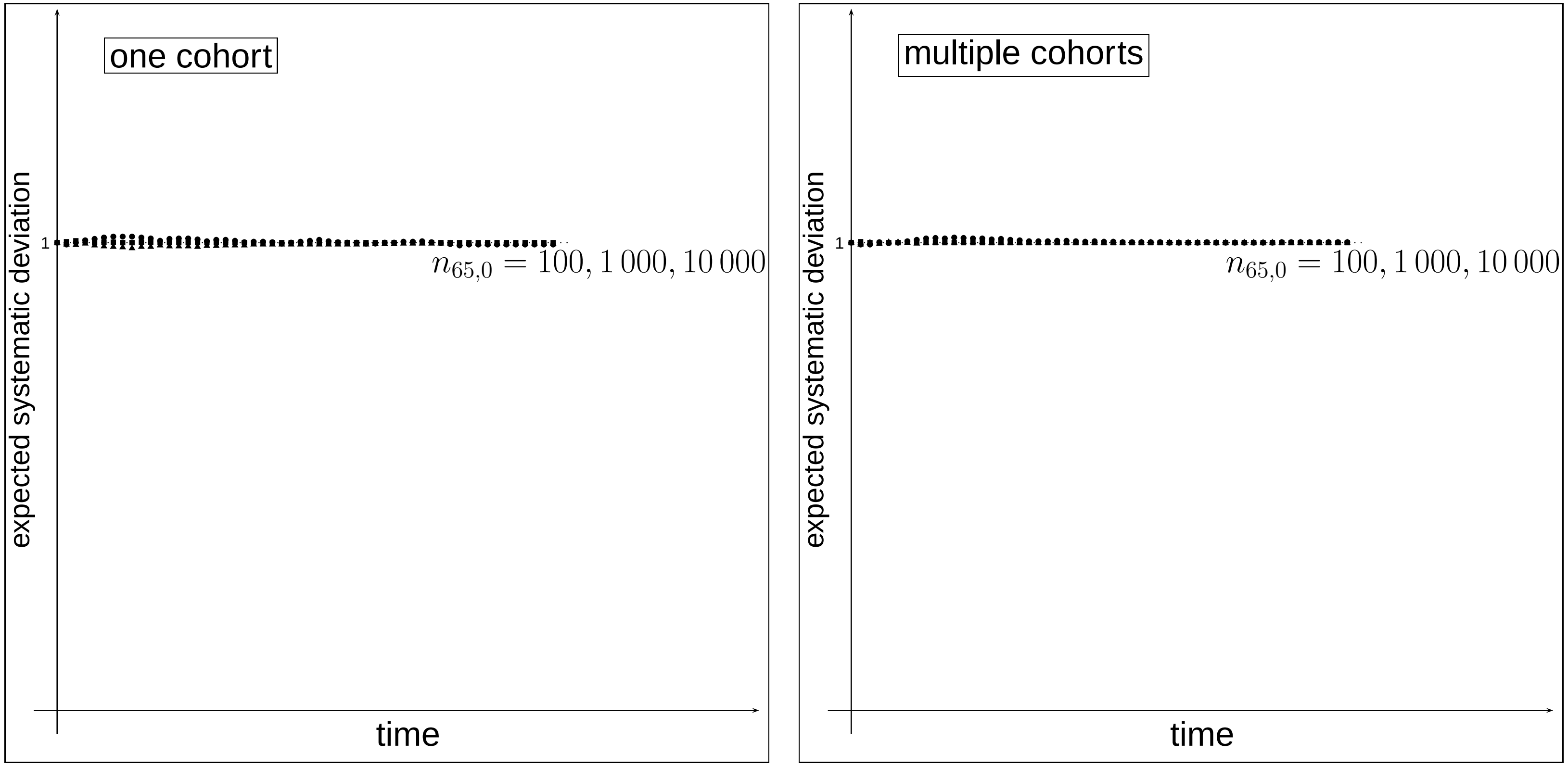


Figure 11 - Expected systematic deviation $\mathbb{E}[Z_t | \{d_s\}_{s=1,2,\dots,t}]$

$\bar{\alpha} = \bar{\beta}$, $\bar{\beta} = 100$; $d_{x,s} = n_{x,s-1}\, q^*_{x,s}$

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

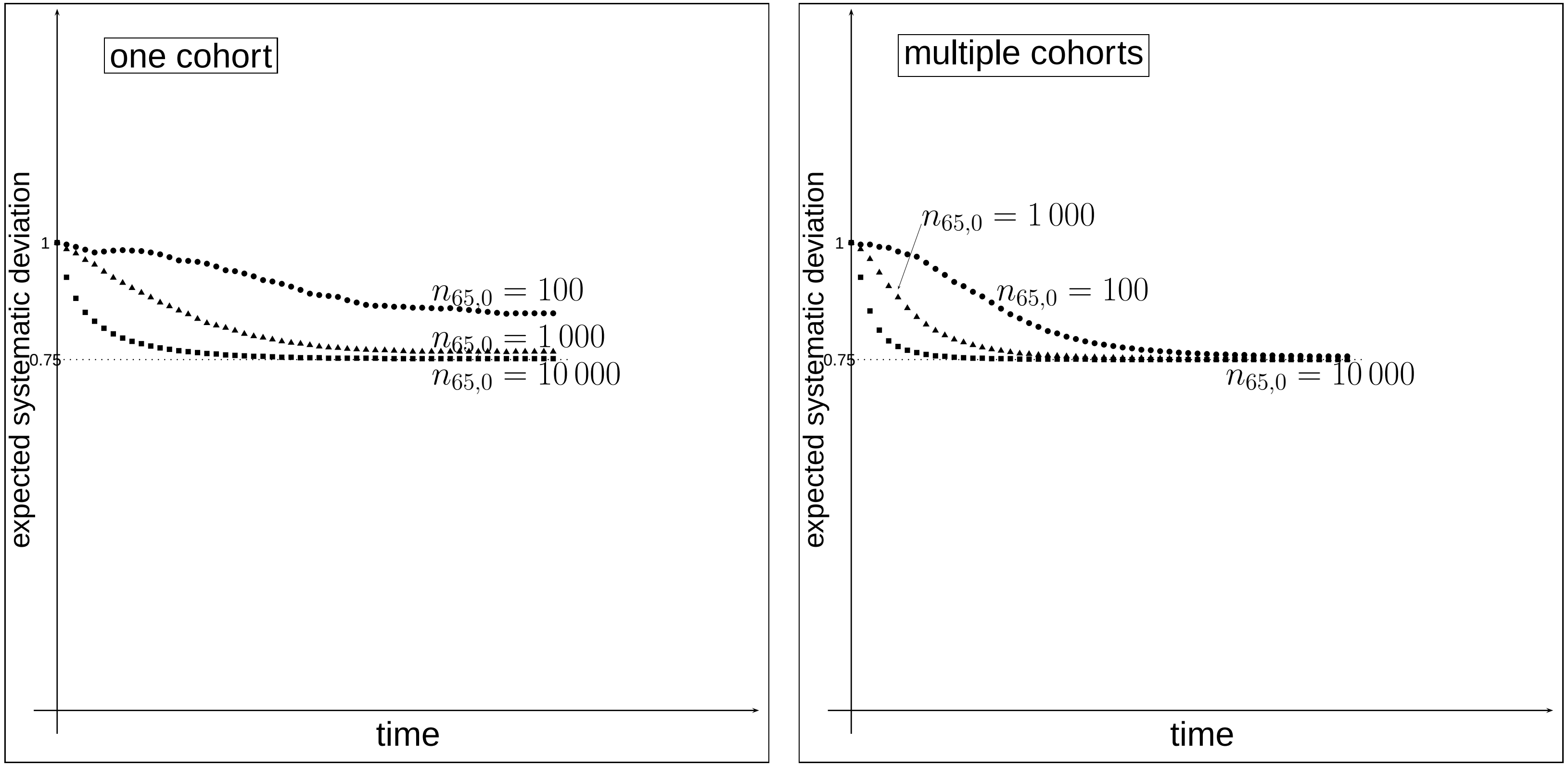


Figure 12 - Expected systematic deviation $\mathbb{E}[Z_t | \{d_s\}_{s=1,2,\ldots,t}]$
$\bar{\alpha} = \bar{\beta}$, $\bar{\beta} = 100$; $d_{x,s} = 0.75\, n_{x,s-1}\, q^*_{x,s}$

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

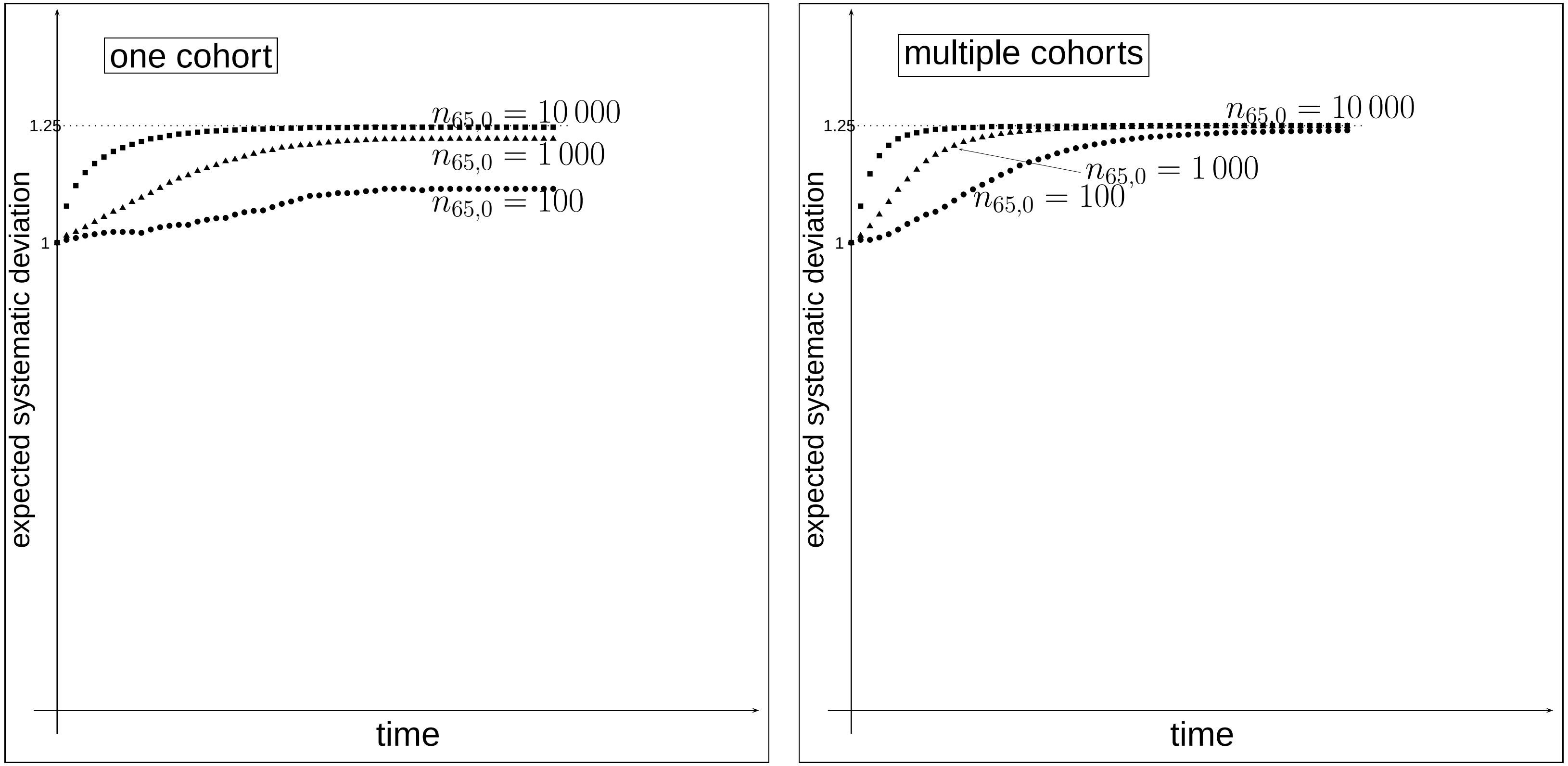


Figure 13 - Expected systematic deviation $\mathbb{E}[Z_t | \{d_s\}_{s=1,2,\dots,t}]$
$\bar{\alpha} = \bar{\beta}$, $\bar{\beta} = 100$; $d_{x,s} = 1.25\, n_{x,s-1}\, q^*_{x,s}$

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

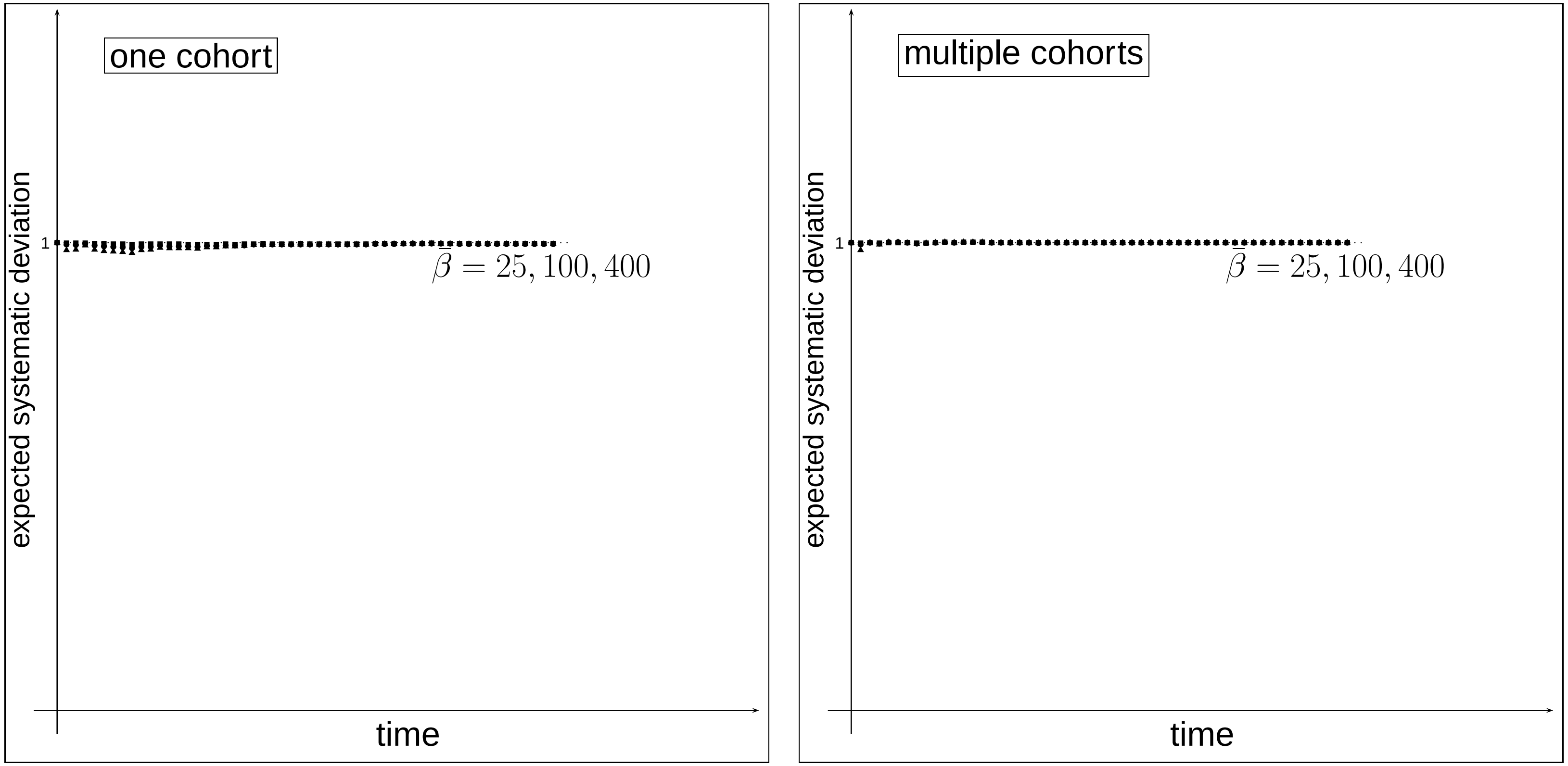


Figure 14 - Expected systematic deviation $\mathbb{E}[Z_t | \{d_s\}_{s=1,2,\dots,t}]$
$\bar{\alpha} = \bar{\beta}$; $n_{65,s} = 1\,000$; $d_{x,s} = n_{x,s-1}\, q^*_{x,s}$

Stochastic mortality: experience-based adjustments (2) *(cont'd)*

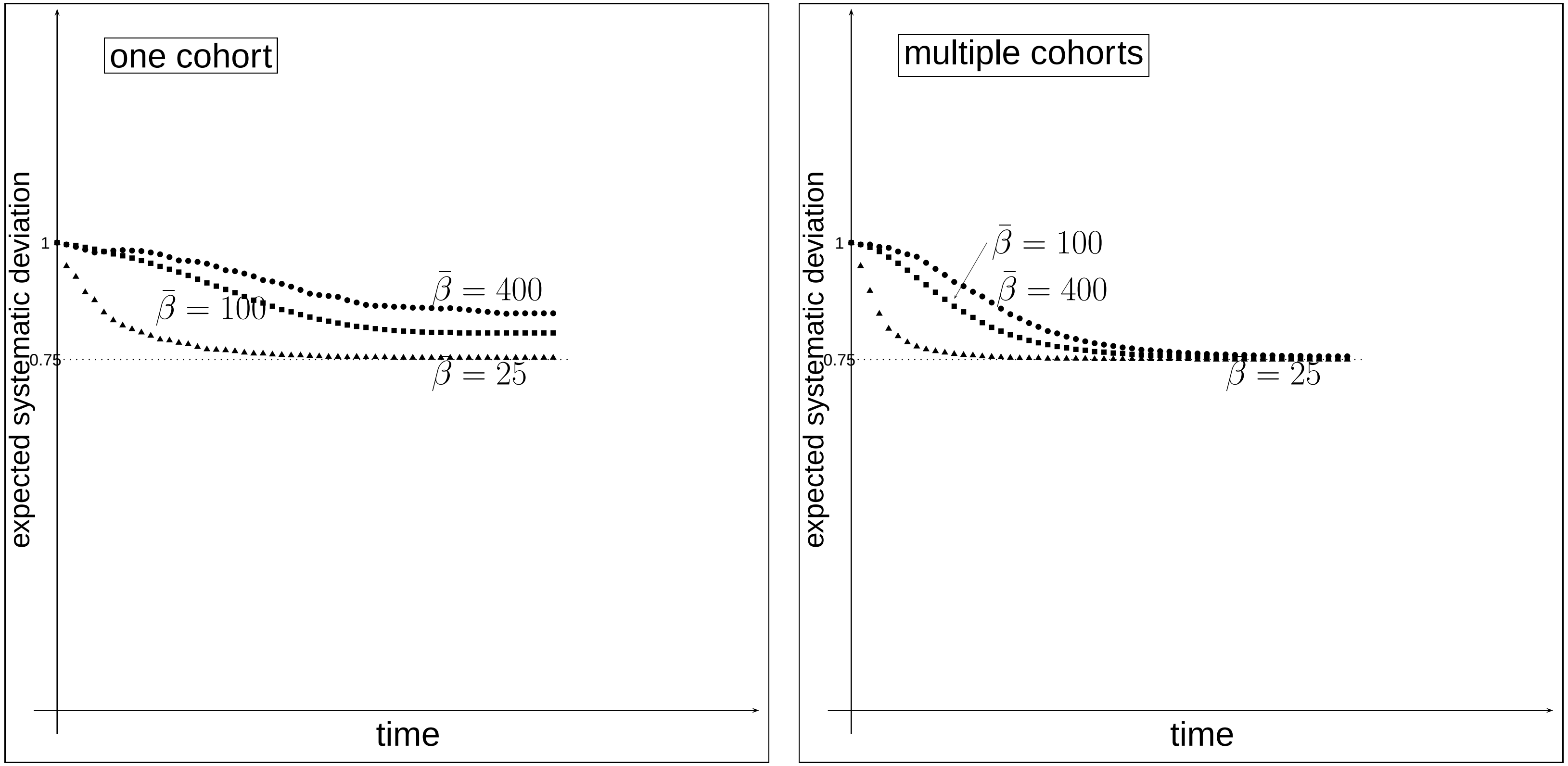


Figure 15 - Expected systematic deviation $\mathbb{E}[Z_t|\{d_s\}_{s=1,2,\dots,t}]$

$\bar\alpha = \bar\beta$; $n_{65,s} = 1\,000$; $d_{x,s} = 0.75\, n_{x,s-1}\, q^*_{x,s}$

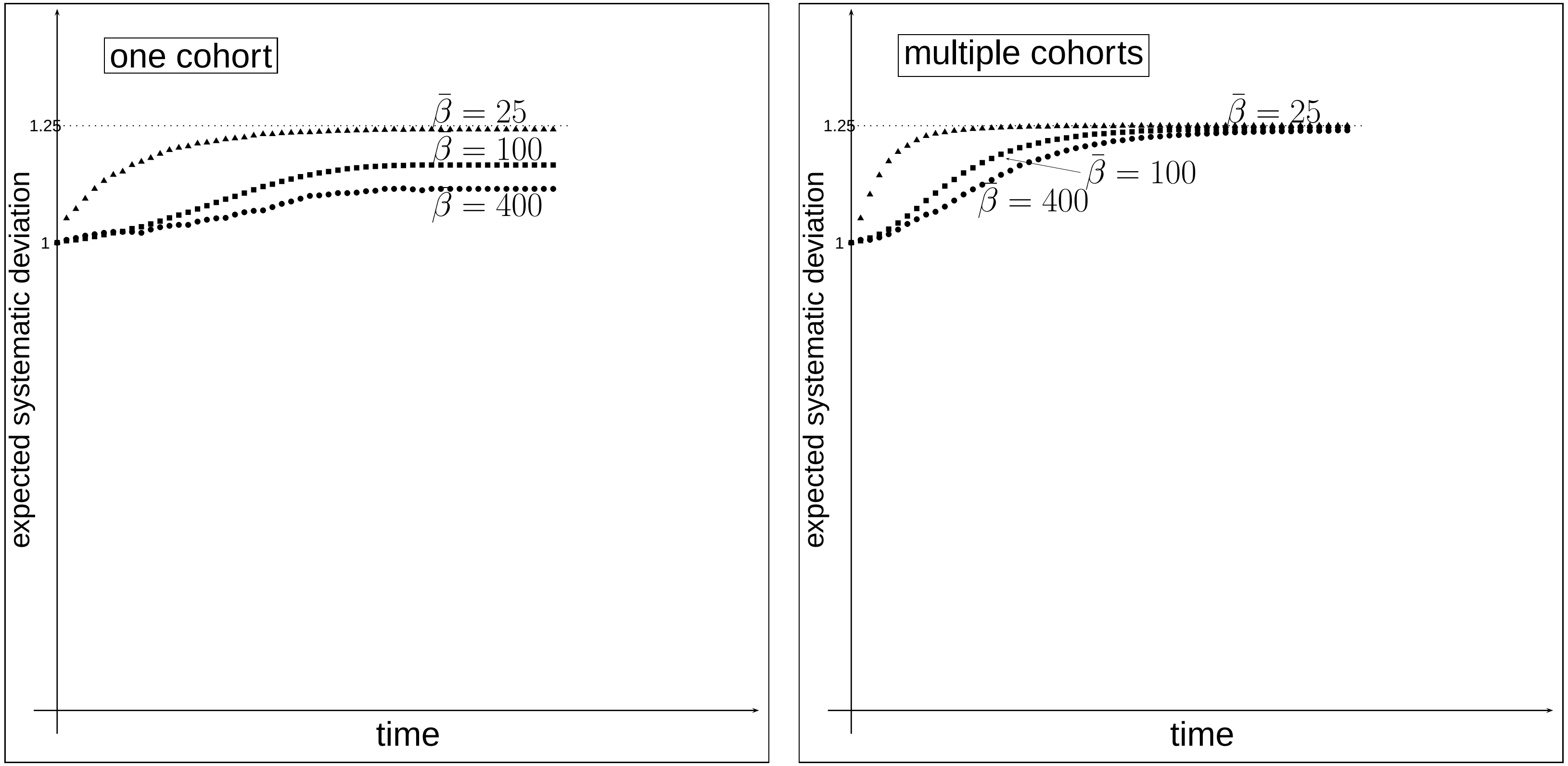


Figure 16 - Expected systematic deviation $\mathbb{E}[Z_t|\{d_s\}_{s=1,2,\dots,t}]$
$\bar\alpha = \bar\beta$; $n_{65,s} = 1\,000$; $d_{x,s} = 1.25\, n_{x,s-1}\, q^*_{x,s}$

CONCLUDING REMARKS

Traditional life insurance mathematics and techniques mainly rely on the calculation of expected values (viz in pricing and reserving), based on given life tables

An appropriate stochastic approach (including mortality / longevity and related trends) is however required because of:

- ▷ the importance of some long-term insurance and pension products
- ▷ the evolving scenarios and the inherent uncertainty
- ▷ a sound assessment of the insurer's risk profile
- ▷ . . .

Starting point of a rigorous stochastic approach to mortality / longevity in life insurance and life annuities $\Rightarrow$ choice of an appropriate probabilistic structure; in particular:

1. a best-estimate life table
2. an inferential procedure for adjustments based on monitoring

Aims of this talk

- ▷ stress the importance of *monitoring* among the "quantitative" steps of the ERM process
- ▷ illustrate (practicable) approaches to inference

Various generalizations of the inferential process could be conceived; for example:

- introduction of cohort specificity
- different probability distributions to express uncertainty
- MCMC methods to approximate posterior distributions
- ...

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Many thanks

for your kind attention !