Monitoring in a risk-management framework: inference from mortality experience in a life annuity portfolio

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UNSW-CSIRO WORKSHOP
Risk: Modelling, Optimization and Inference
(with Applications in Finance, Insurance and Superannuation)

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Agenda

■ Motivation
■ The Enterprise Risk Management framework
■ Monitoring. Experience-based assessments
■ Stochastic mortality: the basics
■ Experience-based adjustment of mortality assumptions (1)
■ Experience-based adjustment of mortality assumptions (2)
■ Concluding remarks

Presentation mostly based on research and teaching material, jointly with Annamaria Olivieri (University of Parma)
MOTIVATION

- Longevity risk: one of the critical issues in managing life annuity portfolios and pension funds
- Need for monitoring mortality trends
  - in populations
  - in specific portfolios and pension funds
- Analysis and choice of risk management actions (reinsurance, longevity swaps, capital allocation, etc.) should be calibrated on updated information
- The monitoring step in the risk management process should include information updating (in particular concerning mortality trends), relying on sound inferential models
THE ENTERPRISE RISK MANAGEMENT (ERM) FRAMEWORK

ERM (in the insurance context):

- Not an alternative to actuarial mathematics and technique
- Provides a unifying point of view: problems, methods, techniques ranging from awareness of risks to management tools
- Reinterpretation of “intuitive” issues
- New issues arising from a comprehensive approach
- A rigorous framework for
  - management practice
  - teaching insurance technique and actuarial mathematics
The Enterprise Risk Management framework (cont’d)

The ERM process

Figure 1 - Steps in the ERM process
We focus on the following steps

- risk identification
- risk assessment
- impact assessment
- monitoring

Risk assessment, impact assessment, monitoring ⇒ quantitative steps in the ERM process
The Enterprise Risk Management framework (cont’d)

\textit{Risk identification}

See, for example: \textit{International Actuarial Association} [2004]

Basic issues

- Risk \textit{sources} (or \textit{causes}) (underwriting, market, operational, etc)

- Risk \textit{components}, in particular:
  - process risk, i.e. the risk of random fluctuations
  - systematic risk, i.e. the risk of systematic deviations (in particular because of uncertainty in model choice and/or parameter estimation)

- Risk \textit{factors}, influencing the severity of impact on portfolio results (portfolio size, policy conditions, etc.)
Risk assessment and impact assessment

In general:

- $X_1, X_2, X_3, \ldots$: random variables representing risks causes
- $c_1, c_2, c_3, \ldots$: values assigned to decision variables
- $Y$: a result chosen to express the impact of risks

Then:

$$Y = \Phi(X_1, X_2, X_3, \ldots; c_1, c_2, c_3, \ldots)$$

Function $\Phi$ should allow (possibly via parameters) for risk factors.

See the following Figure.
The Enterprise Risk Management framework (cont’d)

Figure 2 - Modelling for life insurance and life annuities: a comprehensive approach
How to implement the model?

Ideal target: given

- the joint prob. distribution of \((X_1, X_2, X_3, \ldots)\)

or

- the marginal prob. distributions of \(X_1, X_2, X_3, \ldots\) and correlation assumptions (possibly via copula)

find the probability distribution of \(Y\)

In practice, (almost) impossible to find the probability distribution of \(Y\) via analytical procedures (heavy simplifications usually required)

A wide range of approaches available: from purely deterministic to “completely” stochastic

See the following Figures
## The Enterprise Risk Management framework (cont’d)

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**Figure 3 - Implementing a stochastic model (1-2)**
The Enterprise Risk Management framework  (cont’d)

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- a - analytical
- b - analytical approx
- c - numerical
- d - simulation

stochastic risk assessment and impact assessment, explicitly allowing for uncertainty risk

simulation

stochastic risk assessment and impact assessment, explicitly allowing for stochastic assessment of uncertainty risk

**Figure 4 - Implementing a stochastic model (3-4)**
MONITORING.
EXPERIENCE-BASED ASSESSMENTS

Objectives of the monitoring step (in the ERM process), in particular:

1. checking the effectiveness of the undertaken actions (product design, pricing, reinsurance, swaps, ART, capital allocation, etc.)
2. determining whether changes in the scenario suggest novel solutions
3. adjust probability distributions according to experience
4. .......

Objective 3 ⇒ appropriate modeling structures required, allowing for experience-based adjustments
Monitoring. Experience-based assessments (cont’d)

Figure 5 - A stochastic model allowing for experience-based adjustment (2a)
Monitoring. Experience-based assessments (cont’d)

Figure 6 - A stochastic model allowing for experience-based adjustment (4a)
STOCHASTIC MORTALITY: THE BASICS

*Risks in life insurance, annuities and pensions*

Life insurers and annuity providers take, according to policy conditions (options and guarantees)

- financial risks
- biometric risks (mortality / longevity and disability risks)

Focus on mortality / longevity risks

Risk arising from *individual lifetimes* is a “process risk” (originated by random mortality fluctuations), called *individual mortality / longevity risk*, and can be diversified by increasing the portfolio size or via reinsurance arrangements, i.e. inside the traditional insurance-reinsurance process

Risk arising from *average lifetime in the portfolio* (originated by systematic deviations, in particular because of future unknown mortality trend) is a “systematic risk”, called *aggregate mortality / longevity risk*, and cannot be diversified inside the traditional insurance-reinsurance process
Figure 7 - Simulated number of survivors
(a) random fluctuations ⇒ individual longevity risk
(b) random fluctuations + systematic deviations ⇒ aggregate longevity risk
A basic model

Refer to a cohort initially consisting of \( n_{x_0} \) individuals age \( x_0 \). Define:

\[
T_{x_0}^{(j)} = \text{random lifetime of individual } j \ (j = 1, 2, \ldots, n_{x_0})
\]

\[
N_{x_0+t} = \sum_{j=1}^{n_{x_0}} I\{T_{x_0}^{(j)}>t\}
\]

\[
D_{x_0+t} = N_{x_0+t} - N_{x_0+t+1}
\]

Assume random lifetimes \( T_{x_0}^{(j)}, j = 1, 2, \ldots, n_{x_0} \), are i.i.d., with probability distribution provided by the life table \( \{\ell_x\}_{x=0,1,\ldots,\omega} \)

Then:

\[
N_{x_0+t} \sim \text{Bin}(n_{x_0}, tp_{x_0})
\]

and for \( z > t \),

\[
[N_{x_0+z} \mid n_{x_0+t}] \sim \text{Bin}(n_{x_0+t}, z-tp_{x_0+t})
\]
As regards the numbers $D_{x_0+t}$, conditional on $N_{x_0+t} = n_{x_0+t}$:

$$[D_{x_0+t} \mid n_{x_0+t}] \sim \text{Bin}(n_{x_0+t}, q_{x_0+t})$$

Poisson distribution often adopted as an approximation to the binomial distribution:

$$[D_{x_0+t} \mid n_{x_0+t}] \sim \text{Pois}(n_{x_0+t} q_{x_0+t})$$

with

$$\mathbb{E}[D_{x_0+t} \mid n_{x_0+t}] = n_{x_0+t} q_{x_0+t}$$

(under both the Binomial and the Poisson assumption)

**Example**

$$x_0 = 65; \quad q_x = \frac{G H^x}{1 + G H^x} \quad \text{with} \quad G = 2.005 \times 10^{-6}, \quad H = 1.130$$

See Figures
Figure 8 - Probability distribution of \( \frac{N_{65+t}}{n_{65}} \); \( t = 5, 10, 15 \)
Example of insurance application

Focus on:

\( Y_0^{[P]} \) = random present value at time 0 of the benefits which will be paid by a portfolio of life annuities (individual annual amount \( b \))

\[
Y_0^{[P]} = b \sum_{t=1}^{\omega-x_0} N_{x_0+t} (1 + i)^{-t}
\]

or, equivalently:

\[
Y_0^{[P]} = b \sum_{j=1}^{n_{x_0}} a_{K_{x_0}^{(j)}}
\]

We have:

\[
\mathbb{E} \left[ Y_0^{[P]} \right] = b \sum_{t=1}^{\omega-x_0} \mathbb{E}[N_{x_0+t}] (1+i)^{-t} = b \sum_{t=1}^{\omega-x_0} n_{x_0} \frac{\ell_{x_0+t}}{\ell_{x_0}} (1+i)^{-t} = n_{x_0} b a_{x_0}
\]
Figure 9 - Probability distribution of $\frac{Y_0^{[P]}}{n_{65}}$ ($n_{65} = 100; n_{65} = 1000$)
A simple approach

The approach basically consists of two steps:

1. Choose a set of, say, \( r \) scenarios, in order to express alternative hypotheses about future mortality trend:

\[
\mathcal{H} = \{H_1, H_2, \ldots, H_r\}
\]

- Each scenario: a projected life table (or a survival function, or a force of mortality, etc.)
- Actuarial applications: scenario testing, assessing the range of variation of quantities such as cash flows, profits, portfolio reserves, etc. \( \Rightarrow \) sensitivity analysis
2. Assign non-negative normalized weights to the mortality scenarios ⇒ a probability distribution on the space $\mathcal{H}$:

$$\rho_1, \rho_2, \ldots, \rho_r$$

- Actuarial application: a stochastic approach can be adopted ⇒ unconditional (i.e. non conditional on a particular scenario) variances, percentiles, etc., of the value of future cash flows, profits, etc.

*Example*

See the following Figure
Stochastic mortality: experience-based adjustments (1) *cont’d*

Figure 10 - Conditional and unconditional probability distribution of $\frac{Y_0^{[P]}}{n_{65}}$ ($n_{65} = 100$)

Figure 10 - Conditional and unconditional probability distribution of $\frac{Y_0^{[P]}}{n_{65}}$ ($n_{65} = 100$)
Main feature: a “static” approach

- Uncertainty is expressed just in terms of a set $\mathcal{H}$ of assumptions, and the relevant probability distribution ($\Rightarrow$ which one of the assumptions is the best one for describing the aggregate mortality in the cohort)

- No future shift from such a trend is allowed for in the stochastic model

- Critical aspect: assumptions about the temporal correlation of changes in the probabilities of death are implicitly involved

- Possible mortality shocks are not embedded into the static representation (not a problem when dealing with life annuities)

- Updates of the weights $\rho$ ’s based on experience could be introduced, while keeping the setting as a static one
Modeling with a discrete set of scenarios

See [Olivieri and Pitacco (2002)]

Notation

- \( f(t, y) \): pdf of individual lifetime \( T \), referred to individuals born in year \( y \)
- \( H(y) \): hypothesis about mortality trend for people born in year \( y \)
- family of pdf’s:
  \[
  \{ f(t, y \mid H(y)); H(y) \in \mathcal{H}(y) \}
  \]
- in particular:
  \[
  \{ f(t, y \mid \theta(y)); \theta(y) \in \Theta(y) \}
  \]

where:
- \( \theta(y) \) = vector-valued (in particular, real-valued) parameter
- \( \Theta(y) \) = the parameter space
We focus on the parameterized setting, and refer to a parameter finite space.

For simplicity, we address one generation, hence:

$$\{f(t \mid \theta); \theta \in \Theta\}$$

For the random parameter $\tilde{\theta}$, prior distribution:

$$g(\theta) = Pr[\tilde{\theta} = \theta]$$

Conditional and unconditional expected values can be calculated. In particular, the following result holds:

$$\sqrt{Var[T]} = E[Var[T \mid \tilde{\theta}]] + Var[E[T \mid \tilde{\theta}]]$$

unconditional variance  random fluctuations  systematic deviations
Inference: the model

Refer to a homogeneous set of \( n \) individuals (same generation), considered at time (=age) \( \tau \)

Let

- \( T_h - \tau = \) remaining lifetime of individual \( h \)
- \( T_1, T_2, \ldots, T_n \) iid conditional on any given scenario

Sampling pdf

\[
f_\tau(t \mid \theta) = \begin{cases} 
0 & \text{if } t \leq \tau \\
\frac{f(t \mid \theta)}{\int_\tau^{+\infty} f(u \mid \theta) \, du} & \text{if } t > \tau
\end{cases}
\]

Multivariate sampling pdf

\[
f_\tau(t_1, t_2, \ldots, t_n \mid \theta) = \prod_{h=1}^{n} f_\tau(t_h \mid \theta)
\]
(Prior) predictive pdf (restricted to $[\tau, +\infty)$)

$$f_\tau(t) = \sum_{\theta \in \Theta} f_\tau(t | \theta) g(\theta)$$

Steps of the inferential procedure

- **Observation:** $m$ deaths in the age interval $[\tau, \tau']$, at ages $\underline{x} = (x_1, x_2, \ldots, x_m)$

- **Update the opinion about the possible mortality trend (i.e. about the probability distribution over $\Theta$) ⇒ posterior:**

  $$g(\theta | m, \underline{x}) \propto g(\theta) L(\theta | m, \underline{x})$$

- **Calculate the (posterior) predictive pdf:**

  $$f_\tau(t | m, \underline{x}) = \sum_{\theta \in \Theta} f_\tau(t | \theta) g(\theta | m, \underline{x})$$
Inference: implementation and numerical examples

Weibull law:

\[
f(t \mid \alpha, \beta) = \frac{\alpha}{\beta} \left( \frac{t}{\beta} \right)^{\alpha-1} \exp \left( -\left( \frac{t}{\beta} \right)^{\alpha} \right)
\]

Finite parameter space:

\[
\Theta = \{ (\alpha_i, \beta_j) \}
\]

\[
g(\alpha_i, \beta_j) = \mathbb{P}[\tilde{\alpha} = \alpha_j \land \tilde{\beta} = \beta_j]
\]

Numerical implementation:

\[
\Theta = \{ (\alpha_i, \beta_j); \ i = 1, \ldots, 5, \ j = 1, \ldots, 5 \}
\]

See following Tables
### Stochastic mortality: experience-based adjustments (1) (cont’d)

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**Space $\Theta$:** $\mathbb{E}[T - 65 \mid T > 65; \alpha, \beta]$

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**Space $\Theta$:** $\text{Var}[T - 65 \mid T > 65; \alpha, \beta]$
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Stochastic mortality: experience-based adjustments (1) (cont’d)

Posterior $g$ calculated in several cases ⇒ posterior predictive $f$
⇒ markers of the remaining lifetime

Each case defined by the triplet:

$(\text{prior } g, \text{ actual scenario, simulation})$

where

- prior: $g^{(1)}$ or $g^{(2)}$
- actual scenario (i.e. scenario assumed in the simulation procedure):
  \[ S_h = (\alpha_h, \beta_h); \quad h = 1, 2, \ldots, 5 \]

Note: according to $g^{(1)}$, $S_3 = (\alpha_3, \beta_3)$ is the “best estimate” scenario

- simulation ⇒ number of deaths in the interval $[\tau, \tau'] = [60, 65]$
  ▶ SD = systematic deviations
  ▶ RF = random fluctuations (only)
Stochastic mortality: experience-based adjustments (1) (cont’d)

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<tr>
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Cases \( (g^{(1)}, S_h, SD) \): prior and posterior markers of the remaining lifetime

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<thead>
<tr>
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<th>Prior</th>
<th>Posterior</th>
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<td>( \mathbb{E}[T - 65 \mid T &gt; 65] )</td>
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Cases \( (g^{(2)}, S_h, SD) \): prior and posterior markers of the remaining lifetime
### Stochastic mortality: experience-based adjustments (1) (cont’d)

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*Cases $(g^{(1)}, S_h, RF)$: prior and posterior markers of the remaining lifetime*
STOCHASTIC MORTALITY: EXPERIENCE-BASED ADJUSTMENTS (2)

Focus on longevity risk (both individual and aggregate)

See: Olivieri and Pitacco [2009], Olivieri and Pitacco [2012]

Preliminary ideas

Refer to a portfolio of life annuities (one cohort or multi-cohort)

Assume that:

• a life table, providing a best-estimate of annuitants’ mortality, is available to the insurer

• the insurer has no access to the data sets and the methodology underlying the construction of the life table

• awareness of uncertainty in future mortality trends ⇒ the life table is used as the basic input of a stochastic mortality model
**The model**

We generalize to multi-cohort cases the basic model already described

**Notation:**

- $t_0 =$ starting time of the life annuity portfolio
- $x_0 =$ annuitants’ age at entry
- $t =$ portfolio past duration since time $t_0$; $t = 0, 1, 2, \ldots$
- $D_{x,t} =$ random number of deaths in year $(t - 1, t)$ for those aged $x$ at time $t - 1$
- $N_{x,t} =$ random number of individuals alive age $x$ at time $t$
- $d_{x,t}, n_{x,t} =$ possible (and observed) outcomes of the random variables $D_{x,t}, N_{x,t}$ respectively
Stochastic mortality: experience-based adjustments (2) (cont’d)

Probability distribution (under usual assumptions):

\[ [D_{x,t} \mid q_{x,t}; n_{x,t-1}] \sim \text{Bin}(n_{x,t-1}, q_{x,t}) \]

with \( q_{x,t} = \) assumed probability of death (possibly the best-estimate)

Approximation:

\[ [D_{x,t} \mid q_{x,t}; n_{x,t-1}] \sim \text{Pois}(n_{x,t-1} q_{x,t}) \]

Uncertainty about the mortality trend \( \Rightarrow Q_{x,t} = \) random mortality rate

Modelling approaches:

1. assign a probability distribution to \( Q_{x,t} \)

2. let

\[ Q_{x,t} = q^*_{x,t} Z_{x,t} \]

with \( Z_{x,t} = \) (positive) random adjustment to the best-estimate mortality rate \( q^*_{x,t} \); assign a probability distribution to \( Z_{x,t} \)
Modelling approach 1

Assume:

\[ Q_{x,t} \sim \text{Beta}(a_{x,t}, b_{x,t}) \]

Unconditional distribution of the number of deaths then follows the Beta-Binomial law:

\[
\mathbb{P}[D_{x,t} = d \mid n_{x,t-1}] = \binom{n_{x,t-1}}{d} \frac{\Gamma(a_{x,t} + b_{x,t})}{\Gamma(a_{x,t}) \Gamma(b_{x,t})} \frac{\Gamma(a_{x,t} + d) \Gamma(b_{x,t} + n_{x,t-1} - d)}{\Gamma(a_{x,t} + b_{x,t} + n_{x,t-1})}
\]

where \( \Gamma(\cdot) \) is the incomplete Gamma function

We have:

\[
\mathbb{E}[Q_{x,t}] = \frac{a_{x,t}}{a_{x,t} + b_{x,t}}
\]

\[
\mathbb{E}[D_{x,t} \mid n_{x,t-1}] = n_{x,t-1} \frac{a_{x,t}}{a_{x,t} + b_{x,t}}
\]
Note that:

\[ \mathbb{E}[D_{x,t} \mid q_{x,t}; n_{x,t-1}] = n_{x,t-1} q_{x,t}^* \]

Then

\[ \mathbb{E}[D_{x,t} \mid n_{x,t-1}] \lesssim \mathbb{E}[D_{x,t} \mid q_{x,t}^*; n_{x,t-1}] \]

depending on the comparison between \( q_{x,t}^* \) and \( \mathbb{E}[Q_{x,t}] \)

**Modelling approach 2**

Assume:

\[ Z_{x,t} \sim \text{Gamma}(\alpha_{x,t}, \beta_{x,t}) \]

It turns out:

\[ Q_{x,t} \sim \text{Gamma}\left(\alpha_{x,t}, \frac{\beta_{x,t}}{q_{x,t}^*}\right) \]
Stochastic mortality: experience-based adjustments (2) (cont’d)

and

\[ [D_{x,t} | n_{x,t-1}] \sim \text{NegBin}\left( \alpha_{x,t}, \frac{\theta_{x,t}}{\theta_{x,t} + 1} \right) \] (\(^*\))

with

\[ \theta_{x,t} = \frac{\beta_{x,t}}{n_{x,t-1} q_{x,t}^*} \]

Result (\(^*\)) generalizes the well known Poisson-Gamma structure

We have:

\[ \mathbb{E}[Q_{x,t}] = \frac{\alpha_{x,t}}{\beta_{x,t}} q_{x,t}^* \]

\[ \mathbb{E}[D_{x,t} | n_{x,t-1}] = \frac{\alpha_{x,t}}{\theta_{x,t}} = \frac{\alpha_{x,t}}{\beta_{x,t}} n_{x,t-1} q_{x,t}^* \]
Stochastic mortality: experience-based adjustments (2) (cont’d)

Note that:

\[ \mathbb{E}[D_{x,t}|n_{x,t-1}] \geq \mathbb{E}[D_{x,t}|q_x^*, n_{x,t-1}] \]

depending on the value of \( \frac{\alpha_{x,t}}{\beta_{x,t}} \) (\( \Rightarrow \) systematic deviations in mortality)

Advantages / disadvantages of the approach

- the Gamma distribution does not guarantee that the mortality rate is bounded in \((0, 1)\) (in particular at the oldest ages, when \(q_{x,t}^*\) is high, while \(n_{x,t-1}\) is presumably low)
- the model leads quite naturally to a dynamic setting, through a Bayesian inferential procedure, allowing
  - to account for correlations among the \(Z_{x,t}\)’s
  - to update parameters to experience
- approximation errors at the older ages may become negligible when a portfolio consisting of multiple cohorts is addressed
Rationale of experience-based assessments: a ratio \( \frac{q_{x,t}}{q^*_{x,t}} < 1 \) in year \((t - 1, t)\) (where \( q_{x,t} = \) realized value of \( Q_{x,t} \)) is quite always followed by a ratio \( \frac{q_{x+1,t+1}}{q^*_{x+1,t+1}} < 1 \) (and also \( \frac{q_{x,t+1}}{q^*_{x,t+1}} < 1 \)) in the following year.

**One cohort. The inferential procedure**

Refer to

- one cohort (the model can be applied to the multi-cohort case)
- the Poisson-Gamma model (approach 2)

Assume, for all \( x \) and all \( t \):

\[
Z_{x,t} \sim \text{Gamma} \left( \bar{\alpha}, \bar{\beta} \right)
\]

For example, \( \bar{\alpha}, \bar{\beta} \) such that \( \mathbb{E}[Q_{x,t}] = q^*_{x,t} \) (⇒ one degree of freedom)
It follows:

\[ [D_{x0,1} | n_{x0,0}] \sim \text{NegBin} \left( \bar{\alpha}, \frac{\theta_{x0,1}}{\theta_{x0,1} + 1} \right) \]

where

\[ \theta_{x0,1} = \frac{\bar{\beta}}{n_{x0,0} q^*_x} \]

Let \( d_{x0,1} \) denote the number of deaths observed in year \((0, 1)\)

Then,

\[ n_{x0+1,1} = n_{x0,0} - d_{x0,1} \]

Posterior probability distribution of \( Q_{x0,1} \) conditional on \( D_{x0,1} = d_{x0,1} \):

\[ [Q_{x0,1} | d_{x0,1}] \sim \text{Gamma} \left( \bar{\alpha} + d_{x0,1}, \frac{\bar{\beta}}{q^*_x} + n_{x0,0} \right) \]
Posterior probability distribution of $Z_{x,t}$ conditional on $D_{x_0,1} = d_{x_0,1}$:

$$[Z_{x,t} | d_{x_0,1}] \sim \text{Gamma} \left( \bar{\alpha} + d_{x_0,1}, \bar{\beta} + n_{x_0,0} q_{x_0,1}^* \right)$$

Expected values of $Z_{x,t}$

- **prior**
  $$\mathbb{E}[Z_{x,t}] = \frac{\bar{\alpha}}{\bar{\beta}}$$

- **posterior at time 1**
  $$\mathbb{E}[Z_{x,t} | d_{x_0,1}] = \frac{\bar{\alpha} + d_{x_0,1}}{\bar{\beta} + n_{x_0,0} q_{x_0,1}^*}$$

$$\Rightarrow \mathbb{E}[Z_{x,t} | d_{x_0,1}] \geq \mathbb{E}[Z_{x,t}]$$

depending on the comparison between $d_{x_0,1}$ and the relevant expected value $n_{x_0,0} q_{x_0,1}^*$
Valuations performed at time 1 involving the next year:

\[ Q_{x_0+1,2} | d_{x_0,1} \sim \text{Gamma} \left( \bar{\alpha} + d_{x_0,1}, \frac{\bar{\beta} + n_{x_0,0} q^*_{x_0,1}}{q^*_{x_0+1,2}} \right) \]

and hence

\[ D_{x_0+1,2} | n_{x_0,0}, d_{x_0,1} \sim \text{NegBin} \left( \bar{\alpha} + d_{x_0,1}, \frac{\theta_{x_0+1,2}}{\theta_{x_0+1,2} + 1} \right) \]

with

\[ \theta_{x_0+1,2} = \frac{\bar{\beta} + n_{x_0,0} q^*_{x_0,1}}{n_{x_0+1,1} q^*_{x_0+1,2}} \]

Similar steps at times \( t = 2, 3, \ldots \)
Focus on the expected number of deaths in each year ⇒ at time \( t - 1 \) \((t = 1, 2, \ldots)\) we have:

\[
\mathbb{E}[D_{x_0+t-1,t} | n_{x_0,0}, d_{x_0,1}, d_{x_0+1,2}, \ldots, d_{x_0+t-2,t-1}] = \frac{\bar{\alpha} + \sum_{h=1}^{t-1} d_{x_0+h-1,h}}{\bar{\beta} + \sum_{h=1}^{t-1} n_{x_0+h-1,h-1} q_{x_0+h-1,h}^*} \underbrace{n_{x_0+t-1,t-1} q_{x_0+t-1,t}^*}_{B} \underbrace{A}_{A}
\]

- \( B = \) expected value of \( D_{x_0+t-1,t} \) conditional on best-estimate \( q_{x_0+t-1,t}^* \)
- \( A = \) adjustment coefficient, updated to the observed number of deaths in respect of those expected at the beginning of each year
  - experience consistent with what expected ⇒ coefficient will remain stable in time
  - number of deaths lower than expected ⇒ coefficient will decrease in time
**Numerical findings**

Refer to the expected systematic deviation \( \mathbb{E}[Z_t | \{d_s\}_{s=1,2,...,t}] \)

Choose the best-estimate life table \( \{q^*\} \)

At time 0 set:

- \( \bar{\alpha} = \bar{\beta} \) (\( \Leftarrow \) meaning of the best-estimate life table)
- \( \bar{\beta} = 100 \) (\( \Leftarrow \) expert’s judgment on volatility; see below for alternative choices)

Then:

\[
\begin{align*}
\mathbb{E}[Q_{x,t}] &= q^*_{x,t} \\
\text{Var}[Q_{x,t}] &= \frac{\bar{\alpha}}{(\bar{\beta})^2} (q^*_{x,t})^2 \\
\text{CV}[Q_{x,t}] &= \frac{\sqrt{\text{Var}[Q_{x,t}]}}{\mathbb{E}[Q_{x,t}]} = \frac{1}{\sqrt{\bar{\alpha}}} = 10\%
\end{align*}
\]
Some results

- **Figure 11**: it is assumed \( d_{x,s} = n_{x,s-1} q^*_x \) for \( s = 1, 2, \ldots, t \)
- **Figure 12**: it is assumed \( d_{x,s} = 0.75 n_{x,s-1} q^*_x \Rightarrow \) adjustments
- **Figure 13**: it is assumed \( d_{x,s} = 1.25 n_{x,s-1} q^*_x \Rightarrow \) adjustments
- **Figures 14 - 16**: alternative values for \( \bar{\beta} \) are considered (joint with \( \bar{\alpha} = \bar{\beta} \))
  - \( \bar{\beta} = 100 \Rightarrow \text{CV} [ Q_{x,t} ] = 0.10 \)
  - \( \bar{\beta} = 25 \Rightarrow \text{CV} [ Q_{x,t} ] = 0.20 \)
  - \( \bar{\beta} = 400 \Rightarrow \text{CV} [ Q_{x,t} ] = 0.05 \)

\( \Rightarrow \) effects of the assumed volatility (in terms of \( \text{CV} \)) of the mortality rate, on the expected systematic deviation

Note: numerical results also refer to the multi-cohort case
Stochastic mortality: experience-based adjustments (2) (cont’d)

**Figure 11 - Expected systematic deviation**

\[ \mathbb{E}[Z_t | \{d_s\}_{s=1,2,\ldots,t}] \]

\[ \bar{\alpha} = \bar{\beta}, \bar{\beta} = 100; \quad d_{x,s} = n_{x,s-1} q^*_{x,s} \]
Stochastic mortality: experience-based adjustments (2) (cont’d)

**Figure 12 - Expected systematic deviation**

\[ \mathbb{E}[Z_t | \{d_s\}_{s=1,2,...,t}] \]

\[ \bar{\alpha} = \bar{\beta}, \bar{\beta} = 100; d_{x,s} = 0.75 n_{x,s-1} q_{x,s}^{*} \]
Stochastic mortality: experience-based adjustments (2)  (cont’d)

Figure 13 - Expected systematic deviation

\[ \mathbb{E}[Z_t | \{d_s\}_{s=1,2,\ldots,t}] \]

\[ \bar{\alpha} = \bar{\beta}, \bar{\beta} = 100; d_{x,s} = 1.25 n_{x,s} q_{x,s}^* \]
Stochastic mortality: experience-based adjustments (2) (cont’d)

\[ \bar{\alpha} = \bar{\beta}; n_{65,s} = 1000; d_{x,s} = n_{x,s-1} q'^*_{x,s} \]

**Figure 14 - Expected systematic deviation**

\[ \mathbb{E}[Z_t | \{d_s\}_{s=1,2,\ldots,t}] \]
Stochastic mortality: experience-based adjustments (2)  (cont’d)

Figure 15 - Expected systematic deviation $\mathbb{E}[Z_t | \{d_s\}_{s=1,2,...,t}]$

$\bar{\alpha} = \bar{\beta}; n_{65,s} = 1000; d_{x,s} = 0.75 n_{x,s-1} q^*_x, s$
Figure 16 - Expected systematic deviation  
\[ \mathbb{E}[Z_t | \{d_s\}_{s=1,2,\ldots,t}] \]
\[ \bar{\alpha} = \bar{\beta}; n_{65,s} = 1000; d_{x,s} = 1.25 n_{x,s-1} q^*_{x,s} \]
CONCLUDING REMARKS

Traditional life insurance mathematics and techniques mainly rely on the calculation of expected values (viz in pricing and reserving), based on given life tables. An appropriate stochastic approach (including mortality / longevity and related trends) is however required because of:

- the importance of some long-term insurance and pension products
- the evolving scenarios and the inherent uncertainty
- a sound assessment of the insurer’s risk profile
- ...

Starting point of a rigorous stochastic approach to mortality / longevity in life insurance and life annuities ⇒ choice of an appropriate probabilistic structure; in particular:

1. a best-estimate life table
2. an inferential procedure for adjustments based on monitoring
Concluding remarks (cont’d)

Aims of this talk

- stress the importance of *monitoring* among the “quantitative” steps of the ERM process
- illustrate (practicable) approaches to inference

Various generalizations of the inferential process could be conceived; for example:

- introduction of cohort specificity
- different probability distributions to express uncertainty
- MCMC methods to approximate posterior distributions
- …
References


Many thanks
for your kind attention!