No-Arbitrage ROM Simulation

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Motivation

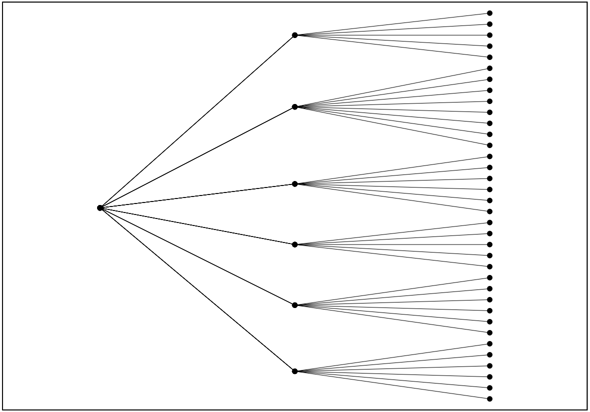
- Discrete samples of multivariate asset return distributions
- For n risky assets with expected (excess) returns $\boldsymbol{\mu}_n$, covariance matrix $\mathbf{S}_n$, and m different states of nature, find

$$\mathbf{X}_{mn} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{bmatrix}$$

such that

$$m^{-1}(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n')'(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n') = \mathbf{S}_n.$$

Motivation



Motivation

- There exist a variety of methods for generating such scenario trees (moment matching, scenario reduction,...)
- Additional requirement for financial applications: samples *must be free of arbitrage*
- Necessary (but not sufficient) condition: $m \geq n$
- Exact matching of the covariance matrix $\mathbf{S}_n$ requires $m \geq n + 1$

Motivation

- Prior to this paper, the standard approach for generating arbitrage-free scenario trees was as follows:
  1. Generate a scenario tree (using any of the available methods)
  2. Check for arbitrage (i.e., solve an LP)
  3. If arbitrage opportunities are found, discard the tree and start again, else: finished
- Problems: Computationally intensive, no guarantee for success for given tree size/branching factor ($\nexists$ theoretical result for required minimum tree size)

Motivation

- Basis for our approach: Ledermann et al. (2011) (ROM simulation – multivariate samples matching pre-specified means and covariances)
- Results: We…
  - extend ROM simulation to ensure arbitrage-free samples,
  - derive analytical bounds to check for arbitrage *ex ante* (without solving an LP),
  - provide insights into the ”geometry of no-arbitrage”.

ROM Simulation

- n assets with expected (excess) returns $\boldsymbol{\mu}_n$ and covariance matrix $\mathbf{S}_n$
- Goal: generate a sample $\mathbf{X}_{mn}$ of m observations on the n random variables such that

$$m^{-1}(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n')'(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n') = \mathbf{S}_n. \tag{1}$$

- $\mathbf{S}_n$ can be decomposed (since pos. semi-def.) into $\mathbf{S}_n = \mathbf{A}_n'\mathbf{A}_n$ (using, e.g., Cholesky decomposition)

ROM Simulation

- Defining

$$\mathbf{L}_{mn} = m^{-1/2}(\mathbf{X}_{mn} - \mathbf{1}_m\boldsymbol{\mu}_n')\mathbf{A}_n^{-1}, \tag{2}$$

solving (1) is equivalent to finding a matrix $\mathbf{L}_{mn}$ satisfying

$$\mathbf{L}_{mn}'\mathbf{L}_{mn} = \mathbf{I}_n \quad \text{with} \quad \mathbf{1}_m'\mathbf{L}_{mn} = \mathbf{0}'. \tag{3}$$

- Ledermann (2011) call solutions to eq. (3) *L matrices*

Mechanics of ROM Simulation

- In general: pre-multiply an L matrix by a permutation matrix and post-multiply this product by any square orthogonal matrix $\mathbf{R}_n$
- Pre-multiplication is primarily for controlling the time-ordering of random samples (not relevant here)
- The basis for our paper is the following simplified version:

$$\mathbf{X}_{mn} = \mathbf{1}_m \boldsymbol{\mu}_n' + \sqrt{m}\mathbf{L}_{mn}\mathbf{R}_n\mathbf{A}_n \tag{4}$$

ROM Simulation

- Since we will frequently need the scaled L matrix with column variance equal to 1, we define $\mathbf{L} = \sqrt{m}\mathbf{L}_{mn}$
- Ledermann et al. (2011) suggest using matrices $\mathbf{R}_n$ representing randomized rotation angles
- Main difference of our extension: *restricted intervals* for random rotation angles

ROM Simulation

- L matrices as defined before have zero mean
- $\mathbf{Y}_{mn} = \mathbf{X}_{mn} - \mathbf{1}_m \boldsymbol{\mu}'_n$ will be important, which can be computed from $\mathbf{L}_{mn}$ using eq. (2):

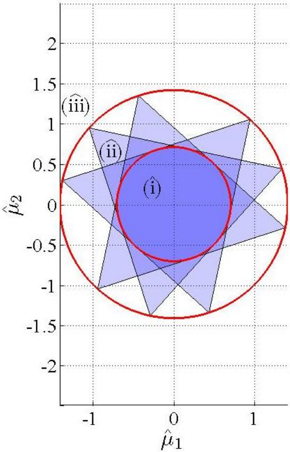
$$\mathbf{Y}_{mn} = \sqrt{m}\mathbf{L}_{mn}\mathbf{A}_n \equiv \mathbf{L}\mathbf{A}_n \tag{5}$$

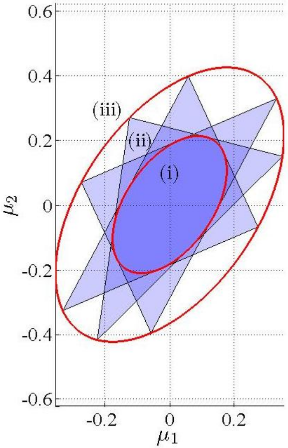
- $\mathbf{Y}_{mn}$ is linked to $\mathbf{L}_{mn}$ by a particular affine transformation $\mathcal{A}(\cdot)$, $\mathbf{Y}_{mn} = \mathcal{A}(\mathbf{L}_{mn})$
- $\mathbf{Y}_{mn}$ can be interpreted as a sample of asset returns with the correct covariance structure $\mathbf{S}_{mn}$ and means of $\mathbf{0_n}$

No-Arbitrage Conditions for ROM Simulation

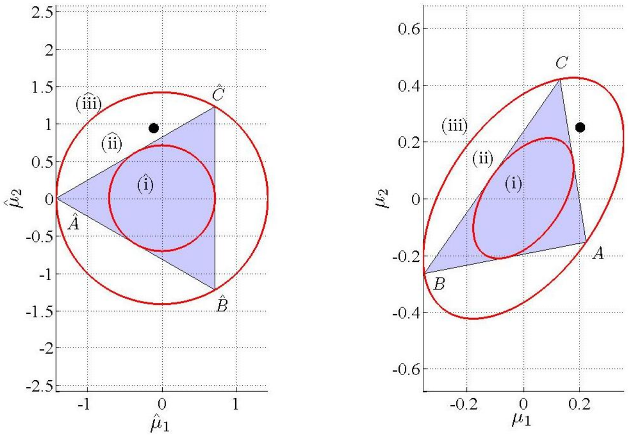
- Geometric interpretation of L matrices: Rows of $-\mathbf{L}_{mn}$ define a simplex (can be constructed deterministically)
- This simplex is regular if $m{=}n{+}1$ ("complete market" with n risky assets and one risk-free asset), and irregular if $m > n{+}1$ ("incomplete market")
- Multiplying the simplex by $\mathbf{R}_n$ rotates the simplex
- Absence of arbitrage means that expected excess returns $\boldsymbol{\mu}_n$ are inside the simplex
- Key insight: $\mathbf{R}_n$ can be chosen judiciously to ensure that $\boldsymbol{\mu}_n$ is inside the simplex

Two-dimensional Case

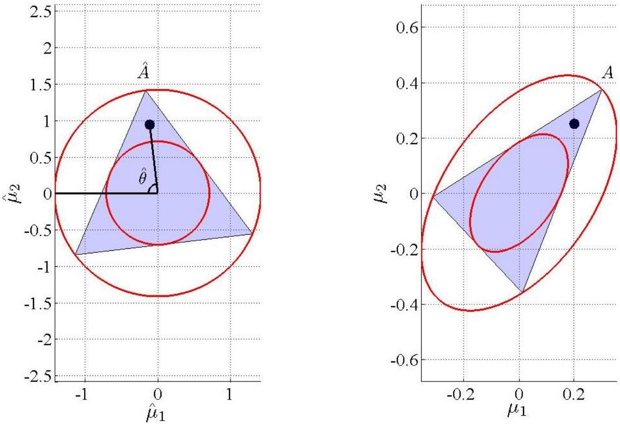




Two-dimensional Case



Two-dimensional Case



Generalization to *n* Dimensions

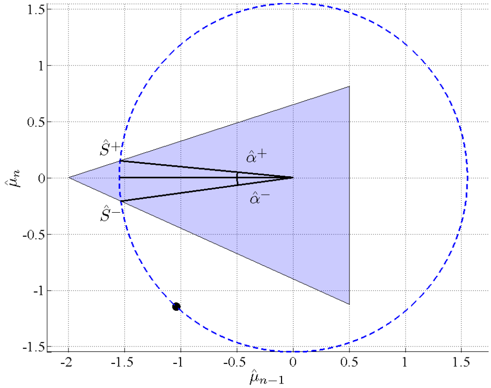
- Equilateral triangle changes to a regular n-simplex
- In- and circumcircles of the triangle become hyperspheres, whose images are hyperellipsoids
- Deterministic construction of the simplex easily generalizes to n dimensions

Rotation in *n* Dimensions

- Concept of rotation of the simplex in the *n*-dimensional case requires a precise definition (Aguilera-Pérez (2004) algorithm)
- Vertex A of simplex and target return define rotation plane
- Rotation occurs around a (*n*-2)-dimensional subspace orthogonal to the rotation plane
- Rotation angle: intersection of the rotation plane with the simplex results in a triangle, which is not regular
- For target returns in region $\widehat{\text{ii}}$, the rotation angle can be sampled from

$$\hat{\gamma} \sim U(\hat{\theta} - \hat{\alpha}^{-}, \hat{\theta} + \hat{\alpha}^{+}) \tag{6}$$

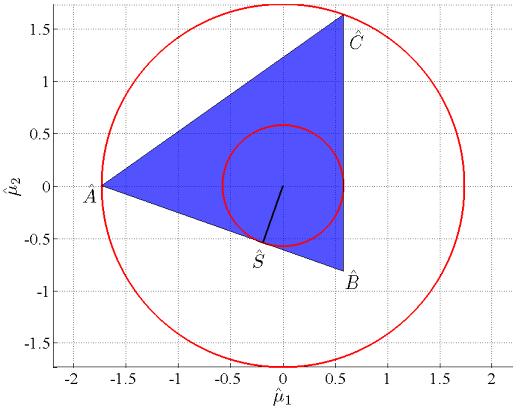
Rotation in n Dimensions



Effects of Increasing the Sample Size m

- What happens to the arbitrage-free regions if $m > n + 1$ (i.e., when increasing the branching factor in scenario trees)?
- $\rightarrow$ n-simplex becomes a polyhedron in n-dimensional space, whose m vertices are given by the rows of $-\mathbf{X}$
- Boundaries between regions $\widehat{(\mathrm{i})}$-$\widehat{(\mathrm{iii})}$ are derived by rotating "extreme" polyhedra

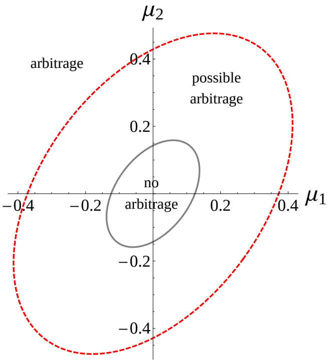
Rotating extreme polyhedra

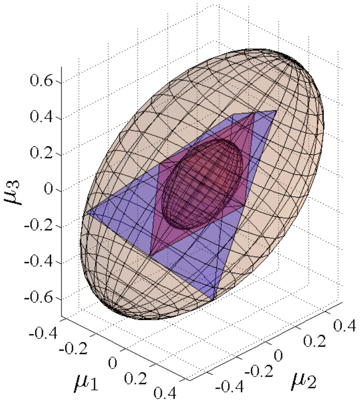


Effects of Increasing the Sample Size m

- Radius of hypersphere, which is guaranteed to be arbitrage-free: $1/\sqrt{m-1}$
- Radius of hypersphere beyond which all scenarios admit arbitrage: $\sqrt{m-1}$
- $\Rightarrow$ Increasing the sample size allows constructing arbitrage-free scenarios for more extreme expected returns
- At the same time, however, increasing m shrinks region $\widehat{(\mathrm{i})}$, which is guaranteed to be free of arbitrage
- (Surprising) Insight: Bounds depend only on the sample size m, but are independent of the number of assets n (!)

(No-)Arbitrage Areas Separated by Analytical Bounds

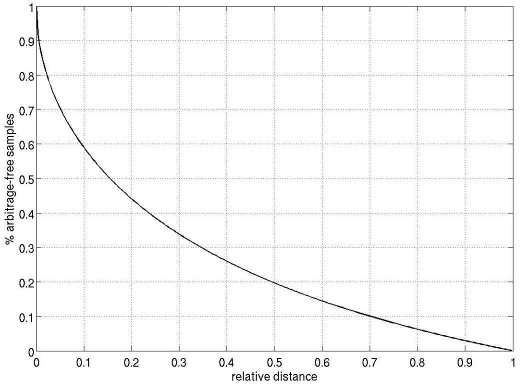




Main Improvements Over Original ROM Simulation

- Analytical bounds classify the problem *ex ante* into 3 areas:
  - Region (i): no-arbitrage is guaranteed. Advantage here: No need to check samples for arbitrage.
  - Region (iii): arbitrage must be present. Advantage here: Known ex ante, together with increase in sample size required to allow for arbitrage-free samples.
  - Region (ii): (possibly frequent) re-sampling is replaced by (one-off) judicious rotation. Size of advantage depends on probability of arriving at arbitrage-free samples when using random rotation angles.

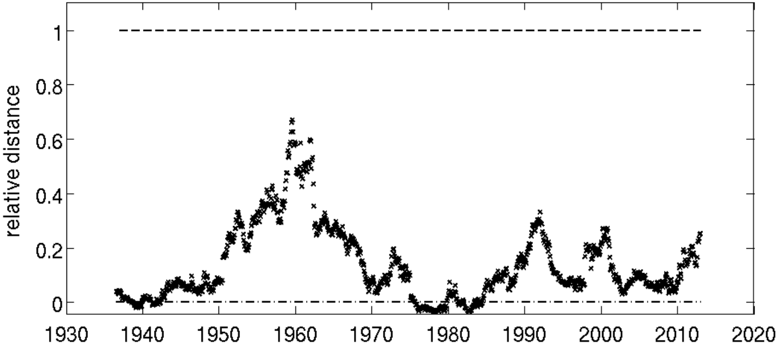
Two-dimensional Case



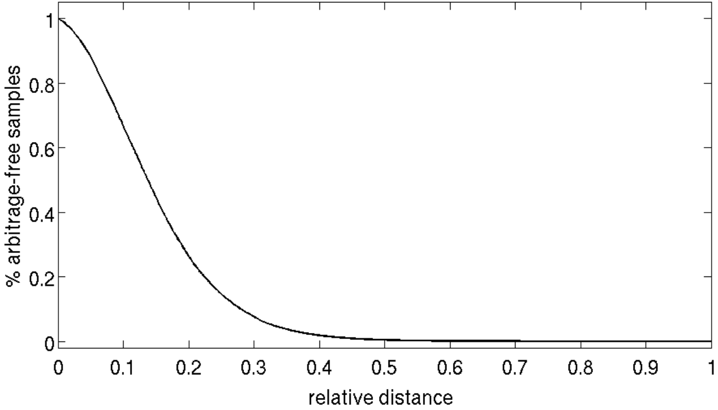
Numerical Example

- Data: ”5 industry portfolios” from K. French’s website (monthly data from 1926-2012)
- Expected returns and covariances are estimated from 10-year rolling windows
- This implies a time-varying Mahalanobis distance
- Using the minimum sample size of m=6, we compute the relative distance between inner and outer ellipsoid
- Depending on this relative distance, we also compute numerically the probability of arriving at arbitrage-free samples when sampling randomly in region (ii)

Numerical Example



Numerical Example



Conclusion

- Extension of original ROM simulation: No-arbitrage ROM simulation algorithm
- If no-arbitrage is theoretically possible: arbitrage-free samples are generated upon the first attempt
- If not: analytical results for bounds provide the minimum sample size to make no-arbitrage possible
- No need for either arbitrage checks or re-sampling
- Retains features of original ROM simulation (i.e., matches first and second moments as well as correlations of multivariate asset return distributions)

Bibliographical data

Geyer, A., M. Hanke, and A. Weissensteiner: No-Arbitrage Bounds for Financial Scenarios. *European Journal of Operational Research*, Vol. 236(2), 657-663.

Geyer, A., M. Hanke, and A. Weissensteiner: No-Arbitrage ROM Simulation. *Journal of Economic Dynamics and Control*, Vol. 45, Aug. 2014, 65-79.