Fair valuation and hedging of contracts under endogenous collateralization

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New challenges

- The financial crisis of 2007-2009 has led to major changes in the operations of financial markets.
- The defaultability of the counterparties became the central problem of financial management.
- The classic paradigm of discounting future cash flows using the risk-free rate is no longer accepted as a viable pricing rule.
- In the presence of funding costs, counterparty credit risk, and collateral (margin account) the classic arbitrage pricing theory no longer applies.
- As a consequence, the analysis of the counterparty credit risk and price formation for collateralized contracts under differential funding costs are currently the most challenging problems in Mathematical Finance.
- A non-linear and asymmetric pricing and hedging paradigm is emerging.

Goals

- To describe trading strategies in the presence of funding costs (multiple yield curves) and margin account (collateral).
- To propose suitable approaches to pricing of financial contracts within this novel framework.
- We focus on one party (dubbed the *hedger*), but the same technique can be used to solve the problem for the *counterparty*.
- The mark-to-market convention for collateral requires that both parties agree in respect of the fair bilateral value of the contract. Hence the actual problem is two-dimensional, rather than one-dimensional.
- The latter issue is especially important in the case of the so-called *endogenous collateral* where we deal with a two-dimensional fully-coupled backward stochastic differential equation (BSDE).

Extended Bergman's (1995) model

- The semimartingale S^i is the price of the ith risky security.
- Cash accounts B^l and B^b for unsecured *lending* and *borrowing* of cash.
- The *collateral* accounts $B^{c,l}$ and $B^{c,b}$ are strictly positive and continuous processes of finite variation.
- A *contract* is a process A representing the *cumulative cash flows*.
- The *collateral process* C with $C_T = 0$ can be represented as

$$C_t = C_t \mathbb{1}_{\{C_t \geq 0\}} + C_t \mathbb{1}_{\{C_t < 0\}} = C_t^+ - C_t^-$$

  where C_t^+ is the cash collateral received at time t by the hedger and C_t^- represents the cash collateral posted by him.
- The process $V(x, \varphi, A, C)$ represents the hedger's wealth.
- The process $V^p(x, \varphi, A, C) = V(x, \varphi, A, C) + C_t$ is the portfolio's value.
- The initial endowment is denoted by x (or rather x_1 and x_2).

Self-financing trading strategy

- For a portfolio $\varphi = (\xi^1, \dots, \xi^d, \psi^l, \psi^b, \eta^b, \eta^l)$, the hedger's wealth process equals

$$V_t(x, \varphi, A, C) = \sum_{i=1}^{d} \xi_t^i S_t^i + \psi_t^l B_t^l + \psi_t^b B_t^b + \eta_t^b B_t^{c,b} + \eta_t^l B_t^{c,l}$$

where $\eta_t^b = -(B_t^{c,b})^{-1} C_t^+$ and $\eta_t^l = (B_t^{c,l})^{-1} C_t^-$.

- A trading strategy (x, φ, A, C) is *self-financing* when the value process

$$V_t^p(x, \varphi, A, C) := \sum_{i=1}^{d} \xi_t^i S_t^i + \psi_t^l B_t^l + \psi_t^b B_t^b$$

satisfies

$$\begin{aligned} V_t^p(x, \varphi, A, C) = x &+ \sum_{i=1}^{d} \int_0^t \xi_u^i \, dS_u^i + \int_0^t \psi_u^l \, dB_u^l + \int_0^t \psi_u^b \, dB_u^b + A_t \\ &+ \int_0^t \eta_u^b \, dB_u^{c,b} + \int_0^t \eta_u^l \, dB_u^{c,l} + C_t. \end{aligned}$$

Funding costs

- We have

$$\psi_t^l = (B_t^l)^{-1}\Big(V_t^p(x,\varphi,A,C) - \sum_{i=1}^d \xi_t^i S_t^i\Big)^+$$

and

$$\psi_t^b = -(B_t^b)^{-1}\Big(V_t^p(x,\varphi,A,C) - \sum_{i=1}^d \xi_t^i S_t^i\Big)^-.$$

- Let $dB_t^l = r_t^l B_t^l\,dt$ and $dB_t^b = r_t^b B_t^b\,dt$ for some processes $0 \le r^l \le r^b$.
- Let $B^{c,l} = B^{c,b} = B^c$ where $dB_t^c = r_t^c B_t^c\,dt$ for some process r^c.
- We define the process F^C

$$F_t^C := \int_0^t \eta_u^b\,dB_u^{c,b} + \int_0^t \eta_u^l\,dB_u^{c,l} = -\int_0^t r_u^c C_u\,du$$

and we denote $A^C := A + C + F^C$.

Dynamics of discounted portfolio's value

Proposition

Let $\widetilde{S}^{i,l}_t := (B^l_t)^{-1} S^i_t$. *The process* $Y^l := (B^l)^{-1} V^p(x, \varphi, A, C)$ *satisfies*

$$dY^l_t = \sum_{i=1}^{d} Z^{l,i}_t \, d\widetilde{S}^{i,l}_t + G_l(t, Y^l_t, Z^l_t)\, dt + (B^l_t)^{-1}\, dA^C_t$$

where $Z^{l,i} = \xi^i$, $i = 1, 2, \ldots, d$ *and the mapping* G_l *equals*

$$G_l(t,y,z) = \sum_{i=1}^{d} r^l_t (B^l_t)^{-1} z^i S^i_t + (B^l_t)^{-1} \bigg(r^l_t \Big(yB^l_t - \sum_{i=1}^{d} z^i S^i_t \Big)^+ - r^b_t \Big(yB^l_t - \sum_{i=1}^{d} z^i S^i_t \Big)^- \bigg) - r^l_t y.$$

Let $\widetilde{S}^{i,b}_t := (B^b_t)^{-1} S^i_t$. *The process* $Y^b := (B^b)^{-1} V^p(x, \varphi, A, C)$ *satisfies*

$$dY^b_t = \sum_{i=1}^{d} Z^{b,i}_t \, d\widetilde{S}^{i,b}_t + G_b(t, Y^b_t, Z^b_t)\, dt + (B^b_t)^{-1}\, dA^C_t$$

where $Z^{b,i} = \xi^i$, $i = 1, 2, \ldots, d$ *and the mapping* G_b *equals*

$$G_b(t,y,z) = \sum_{i=1}^{d} r^b_t (B^b_t)^{-1} z^i S^i_t + (B^b_t)^{-1} \bigg(r^l_t \Big(yB^b_t - \sum_{i=1}^{d} z^i S^i_t \Big)^+ - r^b_t \Big(yB^b_t - \sum_{i=1}^{d} z^i S^i_t \Big)^- \bigg) - r^b_t y.$$

Definition of netted wealth

The concept of the netted wealth is the gateway to study arbitrage issues in our non-linear and asymmetric approach.

Definition

The *netted wealth* $V^{\text{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C)$ is given by

$$V^{\text{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C) := V(y_1, \varphi, A, C) + V(y_2, \widetilde{\varphi}, -A, -C)$$

where $x = y_1 + y_2$ and φ, $\widetilde{\varphi}$ are self-financing trading strategies.

Note that $V_0^{\text{net}}(x, \varphi, A, C) = x$ for any contract (A, C) and any strategy φ.

Definition

A self-financing trading strategy $(y_1, y_2, \varphi, \widetilde{\varphi}, A, C)$ is *admissible* if the discounted netted wealth process $\widetilde{V}^{l,\text{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C) := V^{\text{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C)/B^l$ is bounded from below.

Arbitrage opportunity

Definition

An admissible strategy (x, φ, A, C) is an *arbitrage opportunity for the hedger* with respect to (A, C) whenever

$$\mathbb{P}(V_T^{\text{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C) \geq V_T^0(x)) = 1$$

and

$$\mathbb{P}(V_T^{\text{net}}(y_1, y_2, \varphi, \widetilde{\varphi}, A, C) > V_T^0(x)) > 0$$

where

$$V_t^0(x) := x^+ B_t^l - x^- B_t^b$$

for all $t \in [0, T]$. A model is *arbitrage-free* for the hedger if there is no arbitrage opportunity in regard to any contract (A, C).

Martingale measure and ex-dividend prices

Assumption

There exists a probability measure $\widetilde{\mathbb{P}}^l$ equivalent to $\mathbb{P}$ such that the processes $\widetilde{S}^{i,l}$, $i = 1, 2, \ldots, d$ are $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-local martingales

Proposition

If a martingale measure $\widetilde{\mathbb{P}}^l$ exists and $x_1 \geq 0$, $x_2 \geq 0$ then the model is arbitrage-free for the hedger and for the counterparty.

Definition

Any $\mathcal{G}_t$-measurable random variable for which a replicating strategy for (A, C) over $[t, T]$ exists is called the *hedger's ex-dividend price* at time t for a contract (A, C) and it is denoted by $P^h_t(x_1, A, C)$. Hence for some self-financing strategy φ

$$V_T(V^0_t(x_1) + P^h_t(x_1, A, C), \varphi, A - A_t, C) = V^0_T(x_1).$$

Fair bilateral prices

Definition

For an arbitrary level x_2 of the counterparty's initial endowment and a strategy $\widetilde{\varphi}$ replicating $(-A, -C)$, the *counterparty's ex-dividend price* $P_t^c(x_2, -A, -C)$ at time t for a contract $(-A, -C)$ is implicitly given by the equality

$$V_T(V_t^0(x_2) - P_t^c(x_2, -A, -C), \widetilde{\varphi}, -A + A_t, -C) = V_T^0(x_2).$$

By a *fair bilateral price*, we mean the price level at which no arbitrage opportunity arises for either party. Hence the following definition.

Definition

The $\mathcal{G}_t$-measurable interval

$$\mathcal{R}_t^f(x_1, x_2) := \left[P_t^c(x_2, -A, -C), P_t^h(x_1, A, C)\right]$$

is called the *range of fair bilateral prices* at time t for the contract (A, C).

Bilaterally profitable prices

Definition

Assume that the inequality $P_t^h(x_1, A, C) \neq P_t^c(x_2, -A, -C)$ holds. Then the $\mathcal{G}_t$-measurable interval $\mathcal{R}_t^p(x_1, x_2) := \left[P_t^h(x_1, A, C), P_t^c(x_2, -A, -C)\right]$ is called the *range of bilaterally profitable prices* at time t of an OTC contract (A, C).

Three concepts of arbitrage:

- **(A.1)** The classic definition of an arbitrage opportunity that may arise by trading in primary assets, as in the classic FTAP.
- **(A.2)** An arbitrage opportunity associated with a long hedged position in some contract combined with a short hedged position in the same contract. The contract's price is considered to be exogenously given, but is arbitrary.
- **(A.3)** An arbitrage opportunity related to the fact that the hedger and the counterparty may require different premia to implement their respective (super-)replicating strategies. Here an arbitrage opportunity is simultaneously available to both parties at a *negotiated* OTC price.

Endogenous collateral

- We wish to find out whether the range of fair bilateral prices is non-empty, at least for some classes of contracts (A, C).
- Let C depend on both the hedger's value $V^h := V(x_1, \varphi, A, C)$ and the counterparty's value $V^c := V(x_2, \widetilde{\varphi}, -A, -C)$.
- It is given as follows

$$C_t = q\big(V_t^0(x_1) - V_t^h, V_t^c - V_t^0(x_2)\big) = q(-P_t^h, -P_t^c)$$

where $q : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous function with $q(0,0) = 0$.
- The *convex collateralization* is given by $q(y_1, y_2) = \alpha y_1 + (1-\alpha) y_2$ for some $\alpha \in [0,1]$, so that

$$C_t = \alpha(V_t^0(x) - V_t^h) + (1-\alpha)(V_t^c - V_t^0(x)) = -(\alpha P_t^h + (1-\alpha) P_t^c).$$

- One can also introduce the *haircuts*.

Model assumptions

Assumption

We postulate that:
(i) there exists a probability measure $\widetilde{\mathbb{P}}^l$ equivalent to $\mathbb{P}$ such that $\widetilde{S}^l$ is a continuous, square-integrable, $(\widetilde{\mathbb{P}}^l, \mathbb{G})$-martingale and has the predictable representation property with respect to the filtration $\mathbb{G}$ under $\widetilde{\mathbb{P}}^l$,
(ii) there exists an $\mathbb{R}^{d\times d}$-valued, $\mathbb{G}$-adapted process m^l such that

$$\langle \widetilde{S}^l \rangle_t = \int_0^t m_u^l (m_u^l)^* \, du$$

where the process $m^l(m^l)^*$ is invertible and satisfies $m^l(m^l)^* = \mathbb{S}\sigma\sigma^*\mathbb{S}$ where σ is a d-dimensional square matrix of $\mathbb{G}$-adapted processes satisfying the *ellipticity condition*: there exists a constant $\Lambda > 0$

$$\sum_{i,j=1}^{d} (\sigma_t \sigma_t^*)_{ij} \, a_i a_j \geq \Lambda |a|^2 = \Lambda a^* a, \quad \forall\, a \in \mathbb{R}^d, \, t \in [0,T].$$

The case of hedger's collateral

- Assume first that C depends only on the hedger's value

$$C_t = q(V_t^0(x_1) - V_t^h) = q(-P_t^h)$$

for some Lipschitz continuous function $q : \mathbb{R} \to \mathbb{R}$ such that $q(0) = 0$.
- The price P^c solves the BSDE which depends on the solution P^h and thus the pricing/hedging BSDEs are partially coupled.

Proposition

If $x_1 \geq 0$, $x_2 \geq 0$, then for any contract (A, C) we have for every $t \in [0, T]$

$$P_t^c(x_2, -A, -C) \leq P_t^h(x_1, A, C), \quad \widetilde{\mathbb{P}}^l - \text{a.s.}$$

so that the range of fair bilateral prices $\mathcal{R}_t^f(x_1, x_2)$ is non-empty, $\widetilde{\mathbb{P}}^l - \text{a.s.}$

The range may be empty, in general, if the initial endowments have opposite signs, that is, when $x_1 > 0$ and $x_2 < 0$.

Partially coupled pricing BSDEs

Proposition

Let $x_1 \geq 0$ and $x_2 \geq 0$. The hedger's price equals $P^h := P^h(x_1, A, C) = Y^1$ where (Y^1, Z^1) is the unique solution to the BSDE

$$\begin{cases} dY_t^1 = Z_t^{1,*}\, d\widetilde{S}_t^l + f_l\big(t, x_1, Y_t^1, Z_t^1\big)\, dt + dA_t, \\ Y_T^1 = 0, \end{cases}$$

where

$$\begin{aligned} f_l(t, x_1, y, z) = {} & r_t^l (B_t^l)^{-1} z^* S_t - (B_t^l)^{-1} \sum_{i=1}^{d} r_t^{i,b} (z^i S_t^i)^+ - x_1 B_t^l r_t^l - r_t^c q(-y) \\ & + r_t^l \Big(y + q(-y) + x_1 B_t^l + (B_t^l)^{-1} \sum_{i=1}^{d} (z^i S_t^i)^- \Big)^+ \\ & - r_t^b \Big(y + q(-y) + x_1 B_t^l + (B_t^l)^{-1} \sum_{i=1}^{d} (z^i S_t^i)^- \Big)^- . \end{aligned}$$

Partially coupled pricing BSDEs

Proposition

The counterparty's price equals $P^c := P^c(x_2, -A, -C) = Y^2$ *where* (Y^2, Z^2) *is the unique solution to the BSDE*

$$\begin{cases} dY^2_t = Z^{2,*}_t \, d\widetilde{S}^l_t + g_l\big(t, x_2, Y^2_t, Z^2_t, Y^1_t\big)\, dt + dA_t, \\ Y^2_T = 0, \end{cases}$$

where

$$\begin{aligned} g_l(t, x_2, y, z, Y^1_t) &= r^l_t (B^l_t)^{-1} z^* S_t + (B^l_t)^{-1} \sum_{i=1}^{d} r^{i,b}_t (-z^i S^i_t)^+ + x_2 B^l_t r^l_t - r^c_t q(-Y^1_t) \\ &\quad - r^l_t \Big(-y - q(-Y^1_t) + x_2 B^l_t + (B^l_t)^{-1} \sum_{i=1}^{d} (-z^i S^i_t)^- \Big)^+ \\ &\quad + r^b_t \Big(-y - q(-Y^1_t) + x_2 B^l_t + (B^l_t)^{-1} \sum_{i=1}^{d} (-z^i S^i_t)^- \Big)^- . \end{aligned}$$

Fully coupled pricing BSDEs

- We now consider the case where

$$C_t = q\big(V_t^0(x_1) - V_t^h, V_t^c - V_t^0(x_2)\big) = q(-P_t^h, -P_t^c).$$

- Then the BSDEs for the hedger's and counterparty's prices are fully coupled.

Proposition

Assume that $x_1 \geq 0$ and $x_2 \geq 0$. Then the hedger's and counterparty's prices satisfy $(P^h, P^c)^* = (Y^1, Y^2) = Y$ where (Y, Z) solves the following two-dimensional, fully-coupled BSDE

$$\begin{cases} dY_t = Z_t^* \, d\widetilde{S}_t^l + g\big(t, Y_t, Z_t\big)\, dt + d\overline{A}_t, \\ Y_T = 0, \end{cases}$$

where $g = (g^1, g^2)^*$, $\overline{A} = (A, A)^*$ and ...

Fully coupled pricing BSDEs

Proposition

for all $y = (y_1, y_2)^* \in \mathbb{R}^2$ *and* $z = (z_1, z_2) \in \mathbb{R}^{d\times 2}$,

$$\begin{aligned} g^1(t,y,z) &= r_t^l (B_t^l)^{-1} z_1^* S_t - x_1 B_t^l r_t^l - r_t^c q(-y_1, y_2) \\ &\quad + r_t^l \Big(y_1 + q(-y_1,-y_2) + x_1 B_t^l - (B_t^l)^{-1} z_1^* S_t \Big)^+ \\ &\quad - r_t^b \Big(y_1 + q(-y_1,-y_2) + x_1 B_t^l - (B_t^l)^{-1} z_1^* S_t \Big)^- \end{aligned}$$

and

$$\begin{aligned} g^2(t,y,z) &= r_t^l (B_t^l)^{-1} z_2^* S_t + x_2 B_t^l r_t^l - r_t^c q(-y_1, y_2) \\ &\quad - r_t^l \Big(-y_2 - q(-y_1,-y_2) + x_2 B_t^l + (B_t^l)^{-1} z_2^* S_t \Big)^+ \\ &\quad + r_t^b \Big(-y_2 - q(-y_1,-y_2) + x_2 B_t^l + (B_t^l)^{-1} z_2^* S_t \Big)^- . \end{aligned}$$

Backward stochastic viability property (BSVP)

- Fix $T > 0$ and consider the n-dimensional BSDE

$$Y_t = \eta + \int_t^T h(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s.$$

- The following definition was introduced by Buckdahn, Quincampoix and Rascanu (2000) for a non-empty, closed, convex set of $K \subset \mathbb{R}^n$.

Definition

We say that BSDE has the *backward stochastic viability property* (BSVP) in K if: for any $U \in [0, T]$ and any square-integrable $\eta \in K$ the unique solution (Y, Z) to

$$Y_t = \eta + \int_t^U h(s, Y_s, Z_s)\, ds - \int_t^U Z_s\, dW_s$$

satisfies $Y_t \in K$ for all $t \in [0, U]$, $\mathbb{P}$-a.s.

Multi-dimensional viability theorem

- Let $\Pi_K(y)$ be the projection of a point $y \in \mathbb{R}^n$ onto K.
- Let $d_K(y)$ be the distance between y and K.
- The following result is due to Buckdahn, Quincampoix and Rascanu (2000).

Theorem

Let the generator h of BSDE satisfy the Lipschitz condition and some additional assumptions. Then BSDE has the BSVP in K if and only if for any $t \in [0, T]$, $z \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ such that $d_K^2(\cdot)$ is twice differentiable at y we have

$$4\langle y - \Pi_K(y), h(t, \Pi_K(y), z)\rangle \leq \langle D^2 d_K^2(y)z, z\rangle + M d_K^2(y)$$

where $M > 0$ is a constant independent of (t, y, z).

Comparison theorem for two-dimensional BSDE

Theorem

Consider the two-dimensional BSDE

$$Y_t = \eta + \int_t^T h(s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dW_s.$$

The following statements are equivalent:

(i) for any $U \in [0, T]$ *and* $\eta^1, \eta^2 \in L^2(\Omega, \mathcal{F}_U, \mathbb{P})$ *such that* $\eta^1 \geq \eta^2$*, the unique solution* (Y, Z) *to the BSDE on* $[0, U]$ *satisfies* $Y^1_t \geq Y^2_t$ *for all* $t \in [0, U]$*,*

(ii) there exists a constant M *such that for all* $y, z \in \mathbb{R}^2$

$$\begin{aligned} &-4y_1^-[h^1(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2) - h^2(t, y_1^+ + y_2, y_2, z_1 + z_2, z_2)] \\ &\leq M|y_1^-|^2 + 2z_1^2 \mathbb{1}_{\{y_1 < 0\}}. \end{aligned}$$

Diffusion-type market model

- The risky asset S is governed by the SDE

$$dS_t = \mu(t, S_t)\,dt + \sigma(t, S_t)\,dW_t$$

  where W is a one-dimensional Brownian motion.
- The filtration $\mathbb{G}$ is assumed to be generated by the Brownian motion W.
- The coefficients μ and σ are such that the SDE has a unique strong solution.
- The dividend process equals $A^1_t = \int_0^t \kappa(u, S_u)\,du$.
- We denote

$$a_t := (\sigma(t, S_t))^{-1}\big(\mu(t, S_t) + \kappa(t, S_t) - r^l_t S_t\big).$$

Assumption

We postulate that the processes a, $(\sigma(\cdot, S))^{-1}$ and all interest rates are continuous and the processes a and $(\sigma(\cdot, S))^{-1} S$ are bounded.

Fair prices of European claims

- For a European claim, we have

$$A_t - A_0 = H_T \mathbb{1}_{[T,T]}(t).$$

- Using the comparison theorem for a fully-coupled two-dimensional BSDE, we obtain the following result:

Proposition

Let $x_1 \geq 0$, $x_2 \geq 0$. For any European claim (H_T, C) where $H_T \in L^2(\Omega, \mathcal{F}_T, \widetilde{\mathbb{P}}^l)$ we have for every $t \in [0, T]$

$$P_t^c(x_2, -H_T, -C) \leq P_t^h(x_1, H_T, C), \quad \widetilde{\mathbb{P}}^l - \text{a.s.}$$

so that the range of fair bilateral prices $\mathcal{R}_t^f(x_1, x_2)$ is non-empty.

A similar result holds for any contract (A, C) when $H_{t_i} \in L^2(\Omega, \mathcal{F}_{t_i}, \widetilde{\mathbb{P}}^l)$ and

$$A_t - A_0 = \sum_{i=1}^{l} H_{t_i} \mathbb{1}_{[t_i,T]}(t).$$

Concluding remarks

- We have also studied the case of the market model with funding costs and partial netting.
- However, since each risky asset may have its own funding account, the formulae are rather lengthy.
- Counterparty risk may also be included in the present framework, but new existence and comparison theorems for BSDEs are required to deal with default times.
- For a BSDE approach to mean-variance hedging, see papers by Crépey (2012) and the monograph by Crépey and Bielecki (2014).
- Another interesting concept is the *partial replication* introduced by Burgard and Kjaer (2013). However, its theory is virtually non-existent at present.

Thank you!