GARCH models and their continuous time limits

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Motivation

- The connection between GARCH models and diffusion processes has been well investigated in the last decade (e.g. Nelson (1990, JoE), Duan (1997, JoE), Corradi (2000, JoE), etc)
- There are not many studies which analyze the weak convergence under both physical and risk neutral worlds.
- Duan (1996, unpublished) and Heston and Nandi (2000, RFS) derived the weak limit of Gaussian GARCH models, while Duan (2006, MF) studied the GARCH model based on a Poisson random sum of Gaussian random variables.
- In all three papers, the risk neutralization is based on the locally risk neutral valuation relationship (LRNVR).
- In the Gaussian case, the minimal martingale measure is obtained as the weak limit of the LRNVR.

Our main contributions

1. We study the weak convergence of a general class of non-Gaussian asymmetric GARCH models under both physical and risk-neutral settings.
2. Since the LRNVR cannot be applied in this setting, we derive the continuous time limits based on the extended Girsanov principle (Elliott and Madan (1998, MF) and an exponential affine pricing kernel (Christoffersen, Elkamhi, Feunou and Jacobs (2009, RFS)). The latter is also called the conditional Esscher transform.
3. The convergence of GARCH based European options to diffusion counterparts is numerically tested.

Outline

1. Modelling the underlying under physical measure
2. Modelling the underlying under risk-neutral measures
3. Continuous time diffusion limits
4. Numerical Experiments
5. Conclusions

NGARCH(1,1) model

- $\mathcal{T} = \{t | t = 0, \ldots, T\}$ set of trading dates.
- $Y = \{Y_t\}_{t \in \mathcal{T}} = \{\log S_t\}_{t \in \mathcal{T}}$ has a general NGARCH structure:

$$Y_t - Y_{t-1} = r + \lambda^{(\epsilon)} \sqrt{h_t} - \kappa_{\epsilon_t} \left(\sqrt{h_t} \right) + \sqrt{h_t} \epsilon_t, \tag{1}$$

$$h_t = \alpha_0 + \alpha_1 h_{t-1} \left(\epsilon_{t-1} - \gamma \right)^2 + \beta_1 h_{t-1}. \tag{2}$$

- $\{\epsilon_t\}_{t=0,\ldots,T}$ is a sequence of $\mathcal{F}_{t-1}$ - conditionally i.i.d. random variables with zero mean and unit variance distribution $\mathbf{D}(0,1)$.
- $\kappa_{\epsilon_t}(\cdot)$ is the conditional cumulant generating function of the GARCH innovation.
- The parameter $\lambda^{(\epsilon)}$ is usually interpreted as the market price of ϵ risk.
- The GARCH parameters α_0, α_1, β_1 and γ satisfy the standard non-negativity and stationarity properties.
- γ quantifies the leverage effect.

NGARCH(1,1) approximations under the physical measure

- $\mathcal{T}^{(n)} = \{l | l = k/n, k = 0, 1 \ldots, nT\}$ set of trading dates. The length of each subinterval is $\tau = 1/n$.
- $\left\{Y_l^{(n)}\right\}_{l \in \mathcal{T}^{(n)}} = \left\{\log S_l^{(n)}\right\}_{l \in \mathcal{T}^{(n)}}$ has the following dynamics:

$$
\begin{aligned}
Y_{k\tau}^{(n)} - Y_{(k-1)\tau}^{(n)} &= \left(r + \lambda^{(\epsilon)}\sqrt{h_{k\tau}^{(n)}} - \kappa_{\epsilon_{k\tau}}^{(n)}\left(\sqrt{h_{k\tau}^{(n)}}\right)\right)\tau + \sqrt{\tau h_{k\tau}^{(n)}}\epsilon_{k\tau}^{(n)}, \quad (3) \\
h_{k\tau}^{(n)} - h_{(k-1)\tau}^{(n)} &= \alpha_0(\tau) + \alpha_1(\tau) h_{(k-1)\tau}^{(n)}\left(\epsilon_{(k-1)\tau}^{(n)} - \gamma(\tau)\right)^2 + (\beta_1(\tau) - 1)\, h_{(k-1)\tau}^{(n)}, \quad (4) \\
\epsilon_{k\tau}^{(n)} | \mathcal{F}_{(k-1)\tau}^{(n)} &\sim \mathbf{D}(0,1). \quad (5)
\end{aligned}
$$

- $\left\{\epsilon_{k\tau}^{(n)}\right\}_{k=0,\ldots,nT}$ is a sequence of $\mathcal{F}_{(k-1)\tau}^{(n)}$ - conditionally i.i.d. random variables with zero mean and unit variance distribution $\mathbf{D}$

- Suppose $f_{\epsilon_{k\tau}}^{(n)}(\cdot)$ is the p.d.f. and $\kappa_{\epsilon_{k\tau}}^{(n)}(\cdot)$ the c.g.f. of $\epsilon_{k\tau}^{(n)}$ conditional on $\mathcal{F}_{(k-1)\tau}^{(n)}$ under $P^{(n)}$

$$\kappa_{\epsilon_{k\tau}}^{(n)}(u) := \ln \mathrm{E}^{P^{(n)}}\left[\exp\left(u\epsilon_{k\tau}^{(n)}\right) \mid \mathcal{F}_{(k-1)\tau}^{(n)}\right] < \infty.$$

- The innovations' j^{th} raw moments are:

$$M_j = \mathrm{E}^{P^{(n)}}\left[\left(\epsilon_{k\tau}^{(n)}\right)^j |\mathcal{F}_{(k-1)\tau}^{(n)}\right].$$

- When $\tau = 1$ the model (3)-(5) reduces to the standard asymmetric NGARCH(1,1) model.

The extended Girsanov principle (EGP)

- Identify a change of measure such that discounted asset prices follow the distribution of their martingale component in:

$$\widetilde{S}_{k\tau}^{(n)} = \widetilde{S}_0^{(n)} A_{k\tau}^{(n)} M_{k\tau}^{(n)}.$$

- $A_{k\tau}^{(n)}$ is an $\mathcal{F}_{k\tau}^{(n)}$-predictable process:

$$A_{k\tau}^{(n)} = \prod_{l=1}^{k} \mathrm{E}^{P^{(n)}} \left[\frac{\widetilde{S}_{l\tau}^{(n)}}{\widetilde{S}_{(l-1)\tau}^{(n)}} \middle| \mathcal{F}_{(l-1)\tau}^{(n)} \right].$$

- $M_{k\tau}^{(n)}$ is a positive martingale under $P^{(n)}$ defined by:

$$M_{k\tau}^{(n)} = \frac{\widetilde{S}_{k\tau}^{(n)}}{\widetilde{S}_0^{(n)} A_{k\tau}^{(n)}}.$$

- If we define $W_{k\tau}^{(n)} = M_{k\tau}^{(n)} / M_{(k-1)\tau}^{(n)}$, the Doob decomposition becomes:

$$\widetilde{S}_{k\tau}^{(n)} = \widetilde{S}_{(k-1)\tau}^{(n)} e^{\nu_{k\tau}^{(n)}} W_{k\tau}^{(n)}, \quad k = 0, \ldots, nT.$$

- $\nu_{k\tau}^{(n)}$ is the one period excess discounted return process:

$$\nu_{k\tau}^{(n)} = -r\tau + \log \mathrm{E}^{P^{(n)}} \left[\exp\left(Y_{k\tau}^{(n)} - Y_{(k-1)\tau}^{(n)} \right) \middle| \mathcal{F}_{(k-1)\tau}^{(n)} \right].$$

- In our setting: $\nu_{k\tau}^{(n)} = \lambda^{(\epsilon)} \sqrt{h_{k\tau}^{(n)}} \tau$.
- The EGP is determined by the condition that, under the new measure, discounted asset prices follow the law of their martingale component in the multiplicative Doob decomposition.
- The Radon-Nikodym process $Z_{k\tau}^{(n)}$ is defined via the conditional p.d.f. $g_{W_{k\tau}}^{(n)}(\cdot)$ of $W_{k\tau}^{(n)}$ under $P^{(n)}$:

$$Z_{k\tau}^{(n)} := \frac{dQ_{egp}^{(n)}}{dP^{(n)}}\bigg|_{\mathcal{F}_{k\tau}^{(n)}} = \prod_{l=1}^{k} \frac{g_{W_{l\tau}}^{(n)}\left(\frac{\widetilde{S}_{l\tau}^{(n)}}{\widetilde{S}_{(l-1)\tau}^{(n)}} \right) e^{\nu_{l\tau}^{(n)}}}{g_{W_{l\tau}}^{(n)}\left(e^{-\nu_{l\tau}^{(n)}} \frac{\widetilde{S}_{l\tau}^{(n)}}{\widetilde{S}_{(l-1)\tau}^{(n)}} \right)}.$$

Risk-neutralized dynamics under EGP

Proposition

The risk neutral dynamics of the process $\left\{Y_{k\tau}^{(n)}, h_{k\tau}^{(n)}\right\}_{k=0,\dots,nT}$ *introduced in (3)-(5) with respect to the extended Girsanov risk neutral measure* $Q_{egp}^{(n)}$ *become:*

$$\begin{aligned} Y_{k\tau}^{(n)} - Y_{(k-1)\tau}^{(n)} &= \left(r - \frac{1}{\tau}\kappa_{\epsilon_{k\tau}}^{(n)}\left(\sqrt{\tau h_{k\tau}^{(n)}}\right)\right)\tau + \sqrt{\tau h_{k\tau}^{(n)}}\epsilon_{k\tau}^{*(n)}, \\ h_{k\tau}^{(n)} - h_{(k-1)\tau}^{(n)} &= \alpha_0(\tau) + \alpha_1(\tau) h_{(k-1)\tau}^{(n)}\left(\epsilon_{(k-1)\tau}^{*(n)} - \sqrt{\tau}\varrho_{(k-1)\tau}^{(n)} - \gamma(\tau)\right)^2 + (\beta_1(\tau) - 1)\, h_{(k-1)\tau}^{(n)}, \\ \epsilon_{k\tau}^{*(n)} | \mathcal{F}_{(k-1)\tau}^{(n)} &\sim \mathbf{D}(0,1). \end{aligned}$$

Here, the innovation process $\left\{\epsilon_{k\tau}^{*(n)}\right\}_{k=0,\dots,nT}$ *is a sequence of* $\mathcal{F}_{(k-1)\tau}^{(n)}$*-conditionally uncorrelated, zero mean and unit variance* **D***-distributed random variables under* $Q_{egp}^{(n)}$*, related to the original innovations via the expression:*

$$\epsilon_{k\tau}^{*(n)} = \epsilon_{k\tau}^{(n)} + \sqrt{\tau}\varrho_{k\tau}^{(n)}, \qquad k = 0, \dots, nT,$$

where $\varrho_{k\tau}^{(n)}$ *is given by:*

$$\varrho_{k\tau}^{(n)} = \lambda^{(\epsilon)} + \frac{\frac{1}{\tau}\kappa_{\epsilon_{k\tau}}^{(n)}\left(\sqrt{\tau h_{k\tau}^{(n)}}\right) - \kappa_{\epsilon_{k\tau}}^{(n)}\left(\sqrt{h_{k\tau}^{(n)}}\right)}{\sqrt{h_{k\tau}^{(n)}}}.$$

Conditional Esscher transform (ESS)

- Define the stochastic process $Z^{(n)} = \{Z^{(n)}_{k\tau}\}_{k=0,\dots,nT}$:

$$Z^{(n)}_{k\tau} = \prod_{l=1}^{k} e^{-\sqrt{\tau}\theta^{(n)}_{l\tau}\epsilon^{(n)}_{l\tau} - \kappa^{(n)}_{\epsilon^{(n)}_{l\tau}}\left(-\sqrt{\tau}\theta^{(n)}_{l\tau}\right)}, \quad Z_{0,n} = 1.$$

- $\theta^{(n)} = \{\theta^{(n)}_{k\tau}\}_{k=0,\dots,nT}$ is an $\mathcal{F}^{(n)}$ predictable process satisfying:

$$\mu^{(n)}_{k\tau} + \frac{1}{\tau}\kappa^{(n)}_{\epsilon_{k\tau}}\left(\sqrt{\tau}\left(\sqrt{h^{(n)}_{k\tau}} - \theta^{(n)}_{k\tau}\right)\right) - \frac{1}{\tau}\kappa^{(n)}_{\epsilon_{k\tau}}\left(-\sqrt{\tau}\theta^{(n)}_{k\tau}\right) = r.$$

- Here, $\mu^{(n)}_{k\tau} = r + \lambda^{(\epsilon)}\sqrt{h^{(n)}_{k\tau}} - \kappa^{(n)}_{\epsilon_{k\tau}}\left(\sqrt{h^{(n)}_{k\tau}}\right)$.

- $Z^{(n)}$ is a $P^{(n)}$ martingale and $Z^{(n)}_{nT}$ defines the equivalent martingale measure $Q^{(n)}_{ess}$ by $Z^{(n)}_{nT} = \frac{dQ^{(n)}_{ess}}{dP^{(n)}}$.

Risk-neutralized dynamics under ESS

Proposition

The risk neutral dynamics of $\left\{ Y_{k\tau}^{(n)}, h_{k\tau}^{(n)} \right\}_{k=0,\dots,nT}$ *under the the exponential affine pricing kernel* $Q_{ess}^{(n)}$ *are:*

$$\begin{aligned}
Y_{k\tau}^{(n)} - Y_{(k-1)\tau}^{(n)} &= \left(r + \frac{1}{\sqrt{\tau}} \sqrt{h_{k\tau}^{(n)}} \kappa_{\epsilon_{k\tau}}^{\prime(n)} \left(-\sqrt{\tau}\theta_{k\tau}^{(n)} \right) \right) \tau \\
&+ \kappa_{\epsilon_{k\tau}}^{(n)} \left(-\sqrt{\tau}\theta_{k\tau}^{(n)} \right) - \kappa_{\epsilon_{k\tau}}^{(n)} \left(\sqrt{\tau} \left(\sqrt{h_{k\tau}^{(n)}} - \theta_{k\tau}^{(n)} \right) \right) \\
&+ \sqrt{\tau h_{k\tau}^{(n)}} \sqrt{\kappa_{\epsilon_{k\tau}}^{\prime\prime(n)} \left(-\sqrt{\tau}\theta_{k\tau}^{(n)} \right)} \epsilon_{k\tau}^{*(n)}, \\
h_{k\tau}^{(n)} - h_{(k-1)\tau}^{(n)} &= \alpha_0(\tau) + \alpha_1(\tau) h_{k\tau}^{(n)} \left(\sqrt{\kappa_{\epsilon_{k\tau}}^{\prime\prime(n)} \left(-\sqrt{\tau}\theta_{k\tau}^{(n)} \right)} \epsilon_{k\tau}^{*(n)} + \kappa_{\epsilon_{k\tau}}^{\prime(n)} \left(-\sqrt{\tau}\theta_{k\tau}^{(n)} \right) - \gamma(\tau) \right)^2 \\
&+ \left(\beta_1(\tau) - 1 \right) h_{k\tau}^{(n)}, \\
\epsilon_{k\tau}^{*(n)} | \mathcal{F}_{(k-1)\tau}^{(n)} &\sim \mathbf{D}^*(0,1).
\end{aligned}$$

The innovation $\left\{ \epsilon_{k\tau}^{*(n)} \right\}_{k=0,\dots,nT}$ *is a sequence of* $\mathcal{F}_{(k-1)\tau}^{(n)}$*-conditionally* $\mathbf{D}^*(0,1)$*-distributed under* $Q_{ess}^{(n)}$*:*

$$\epsilon_{k\tau}^{*(n)} = \frac{\epsilon_{k\tau}^{(n)} - \kappa_{\epsilon_{k\tau}}^{\prime(n)} \left(-\sqrt{\tau}\theta_{k\tau}^{(n)} \right)}{\sqrt{\kappa_{\epsilon_{k\tau}}^{\prime\prime(n)} \left(-\sqrt{\tau}\theta_{k\tau}^{(n)} \right)}}.$$

Gaussian Innovations

- When the innovation noise is Gaussian, both propositions lead to the same risk neutralized dynamics obtained via the local risk neutral valuation principle (LRNVP).
- $\varrho_{k\tau}^{(n)} = \theta_{k\tau}^{(n)} = \lambda^{(\epsilon)}$ for any $k = 0, \ldots, nT$.
- When $\tau = 1$, these equations reduces to the GARCH option pricing of Duan (1995) with a leverage effect.
- The Radon-Nykodym derivative has the following explicit form of a discretized Girsanov change of measure in continuous time corresponding to a market price of risk $\lambda^{(\epsilon)}$:

$$\left.\frac{dQ^{(n)}}{dP^{(n)}}\right|_{\mathcal{F}_{k\tau}^{(n)}} = \exp\left(\sum_{l=1}^{k}\left(-\sqrt{\tau}\lambda^{(\epsilon)}\epsilon_{k\tau}^{(n)} - \frac{1}{2}\tau\left(\lambda^{(\epsilon)}\right)^2\right)\right).$$

Construction and constraints

- The right continuous with left limit (cadlag) extension of the proposed discrete time process is defined by:

$$\left\{ Y_t^{(n)}, h_t^{(n)} \right\}_{k\tau \le t < (k+1)\tau} := \left\{ Y_{k\tau}^{(n)}, h_{k\tau}^{(n)} \right\}, \quad k = 0, \ldots, nT.$$

- Define $\left\{ \mathcal{F}_t^{(n)} \right\}_{k\tau \le t < (k+1)\tau} := \mathcal{F}_{k\tau}^{(n)}$, $k = 0, \ldots, nT$, and write $\mathcal{F}_t^{(n),h} := \mathcal{F}_t^{(n)} \bigcup \left\{ h_t^{(n)} = h \right\}$.

- **Parametric Constraints** (see Nelson, 1990)

$$\lim_{\tau \to 0} \frac{\alpha_0(\tau)}{\tau} = \omega_0, \quad \lim_{\tau \to 0} \frac{\alpha_1(\tau)(1 + \gamma^2(\tau)) + \beta_1(\tau) - 1}{\tau} = -\omega_1,$$

$$\lim_{\tau \to 0} \frac{\alpha_1^2(\tau)}{\tau} = \omega_2, \quad \lim_{\tau \to 0} \gamma(\tau) = \omega_3.$$

Stochastic volatility limit under P

Proposition

Assume the above parameter conditions hold. Then, as τ approaches zero, the process $\left\{Y_t^{(n)}, h_t^{(n)}\right\}$ converges weakly to a bivariate diffusion $\left(Y_t, \sigma_t^2\right)$ which satisfies the following stochastic differential equation:

$$
\begin{aligned}
dY_t &= \left(r + \lambda^{(\epsilon)} \sqrt{h_t}_t - \kappa_t \left(\sqrt{h_t}\right)\right) dt + \sqrt{h_t} dB_t^{(1)}, \\
dh_t &= \left(\omega_0 - \omega_1 h_t\right) dt + \sqrt{\omega_2}(M_3 - 2\omega_3) h_t dB_t^{(1)} \\
&+ \sqrt{\omega_2}\sqrt{M_4 - M_3^2 - 1} h_t dB_t^{(2)}.
\end{aligned}
$$

Here, $B_t^{(1)}$ and $B_t^{(2)}$ are two independent Brownian motions on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\dots,T]}, P\right)$.

Comments

- In the case of Gaussian innovations the above coincides with the standard asymmetric GARCH diffusion limit of Duan (1997).
- The use of a non-Gaussian distribution for the underlying discrete process does not alter the Hull-White structure of the variance equation.
- The diffusion coefficient of the stochastic volatility dynamics incorporates the skewness and the kurtosis of the distribution.

Proposition

Under the same parametric conditions, the risk neutral processes under $Q_{ess}^{(n)}$ converge weakly to the same bivariate diffusion limit given below:

$$\begin{aligned} dY_t &= \left(r - \frac{1}{2}h_t\right)dt + \sqrt{h_t}dB_t^{*(1)}, \\ dh_t &= \left(\omega_0 - \left(\omega_1 + \sqrt{\omega_2}(M_3 - 2\omega_3)\nu_t^{(1)} + \sqrt{\omega_2}\nu_t^{(2)}\sqrt{M_4 - M_3^2 - 1}\right)h_t\right)dt \\ &+ \sqrt{\omega_2}\left(M_3 - 2\omega_3\right)h_t dB_t^{*(1)} + \sqrt{\omega_2}\sqrt{M_4 - M_3^2 - 1}h_t dB_t^{*(2)}. \end{aligned}$$

Here $B_t^{*(1)}$ and $B_t^{*(2)}$ are two independent Brownian motions under $Q_{ess}^{(n)}$:

$$B_t^{*(1)} = B_t^{(1)} + \int_0^t \nu_s^{(1)} ds, \quad B_t^{*(2)} = B_t^{(2)} + \int_0^t \nu_s^{(2)} ds,$$

and the market prices of $B_t^{(1)}$ and $B_t^{(2)}$ risk are given by:

$$\nu_t^{(1)} = \lambda^{(\epsilon)} + \frac{\frac{1}{2}h_t - \kappa_t(\sqrt{h_t})}{\sqrt{h_t}}, \quad \nu_t^{(2)} = 0.$$

Proposition

Under the same parametric conditions, the risk neutral processes under $Q_{egp}^{(n)}$ converge weakly to the same bivariate diffusion limit given below:

$$\begin{aligned} dY_t &= \left(r - \frac{1}{2}h_t\right)dt + \sqrt{h_t}dB_t^{*(1)}, \\ dh_t &= \left(\omega_0 - \left(\omega_1 + \sqrt{\omega_2}(M_3 - 2\omega_3)\nu_t^{(1)} + \sqrt{\omega_2}\nu_t^{(2)}\sqrt{M_4 - M_3^2 - 1}\right)h_t\right)dt \\ &+ \sqrt{\omega_2}\left(M_3 - 2\omega_3\right)h_t dB_t^{*(1)} + \sqrt{\omega_2}\sqrt{M_4 - M_3^2 - 1}h_t dB_t^{*(2)}. \end{aligned}$$

Here $B_t^{*(1)}$ and $B_t^{*(2)}$ are two independent Brownian motions under $Q_{egp}^{(n)}$:

$$B_t^{*(1)} = B_t^{(1)} + \int_0^t \nu_s^{(1)} ds, \quad B_t^{*(2)} = B_t^{(2)} + \int_0^t \nu_s^{(2)} ds,$$

and the market prices of $B_t^{(1)}$ and $B_t^{(2)}$ risk are given by:

$$\nu_t^{(1)} = \lambda^{(\epsilon)} + \frac{\frac{1}{2}h_t - \kappa_t(\sqrt{h_t})}{\sqrt{h_t}}, \quad \nu_t^{(2)} = -\nu_t^{(1)}\frac{M_3}{\sqrt{M_4 - M_3^2 - 1}}.$$

Comments

These results can be viewed as extensions of Duan's (1996) convergence theorem of locally risk neutralized Gaussian GARCH models obtained via LRNVR:

- *Gaussian innovations:*
  - The price of risk processes are $\nu_t^{(1)} = \lambda^{(\epsilon)}$ and $\nu_t^{(2)} = 0$ (since $M_3 = 0$ in the EGP case) for both risk neutral measures.
  - The variance equation reduces to the well-known GARCH diffusion process obtained by applying the minimal martingale measure (MMM) to the SV model under P.

$$dh_t = \left(\omega_0 - \left(\omega_1 - 2\sqrt{\omega_2}\omega_3\right) h_t\right) dt - 2\sqrt{\omega_2}\omega_3 h_t dB_t^{*(1)} + \sqrt{2\omega_2} h_t dB_t^{*(2)}.$$

- *Non-Gaussian innovations:*
  - For the conditional Esscher transform, we obtain the MMM as the weak limit.
  - For the extended Girsanov Principle, we obtain the MMM only if we model the underlying with symmetric distributions, since $\nu_t^{(2)} = 0$ whenever $M_3 = 0$. However, skewness plays an important role here as it induces a non-zero market price of non-hedgeable risk in the continuous time limit.

- For both cases, the resulting variance process does not have a Hull-White structure since the drift is not a linear function of h_t.
- It has a non-linear dependence through the cumulant generating function of the GARCH noise.
- Using a Taylor expansion of the second order or of the fourth order we obtain:

$$\nu_t^{(1)} = \lambda^{(\epsilon)}, \quad \nu_t^{(1)} = \lambda^{(\epsilon)} - \frac{1}{6} M_3 h_t - \frac{1}{24}\sqrt{h_t^3}(M_4 - 3).$$

- The resulting return equation ensures that the discounted asset price is a local martingale under Q.
- $\tilde{S}_t$ is a true martingale is equivalent to having a non-positive correlation between the asset return and its variance provided that the market price of $B_t^{(1)}$ risk $\nu_t^{(1)}$ is bounded.

$$\mathrm{Cov}(dY_t, dh_t) = \sqrt{\omega_2}\,(M_3 - 2\omega_3)\,\sqrt{h_t^3}.$$

NGARCH(1,1) innovations, estimation and simulation

- We compute differences between NGARCH(1,1) based on Gaussian and NIG innovations and their diffusion limits option prices.
- For NIG we denote $\epsilon_{k\tau}^{(n)} \sim \mathbf{NIG}(k, a, s, \ell)$, with cumulant generating function given by:

$$\kappa_{\epsilon_{k\tau}}^{(n)}(z) = z\ell + \left(\sqrt{k^2 - a^2} - \sqrt{k^2 - (a + zs)^2}\right).$$

- The model parameters are obtained by fitting the NGARCH(1,1) model from (3)-(4) with $\tau = 1$ via maximum likelihood estimation (MLE). We use daily log-returns of the S&P 500 index from January 2nd, 1988 to April 17th, 2002.
- **NGARCH(1,1) model with Gaussian innovations (NGARCH)**:

$$\alpha_0(1) = 9.9411{\cdot}10^{-7}, \quad \alpha_1(1) = 0.0417, \quad \beta_1(1) = 0.9176, \quad \gamma(1) = 0.8639, \quad \lambda = 0.0414.$$

- **NGARCH(1,1) model with NIG innovations (NIG-NGARCH)**:

$$\alpha_0(1) = 8.6650{\cdot}10^{-7}, \quad \alpha_1(1) = 0.0479, \quad \beta_1(1) = 0.9096, \quad \gamma(1) = 0.8601, \quad \lambda = 0.0419.$$

- Additionally, the NIG invariant parameters are $k = 1.7190$ and $a = -0.1869$.

- We compute the prices associated with European put options based on the daily GARCH process as well as its diffusion limits.
- GARCH prices are computed as discounted expected payoffs under the risk neutral measure, since there are no closed form solutions
- Since there are no closed form solutions, prices are obtained as the average of 120 Monte Carlo simulations involving $100,000$ paths each.
- Diffusion prices are computed based on sample paths simulated using an Euler discretization of 1,024 steps per day, and the model parameters are those induced by the corresponding GARCH processes.
- We use the empirical martingale simulation (EMS) of Duan and Simonato (1998).

Gaussian NGARCH and its diffusion limit Put option prices

Gaussian-NGARCH (daily frequency) and SV Prices for European Put Options $S_0 = 100$						
Maturity	**Model**	Strike=80	Strike=90	Strike=100	Strike=110	Strike=120
7	**NGARCH**	0.000	0.001	1.144	9.979	19.976
		(0.000)	(0.000)	(0.003)	(0.000)	(0.000)
	SV	0.000	0.001	1.152	9.978	19.976
		(0.000)	(0.000)	(0.003)	(0.000)	(0.000)
21	**NGARCH**	0.001	0.082	1.964	9.967	19.930
		(0.000)	(0.002)	(0.005)	(0.001)	(0.000)
	SV	0.001	0.078	1.979	9.958	19.929
		(0.000)	(0.002)	(0.005)	(0.001)	(0.000)
63	**NGARCH**	0.101	0.680	3.356	10.243	19.804
		(0.003)	(0.007)	(0.009)	(0.004)	(0.001)
	SV	0.094	0.682	3.378	10.230	19.798
		(0.003)	(0.007)	(0.009)	(0.004)	(0.001)
126	**NGARCH**	0.460	1.594	4.692	10.919	19.755
		(0.007)	(0.011)	(0.013)	(0.009)	(0.004)
	SV	0.453	1.606	4.718	10.919	19.739
		(0.008)	(0.012)	(0.013)	(0.009)	(0.003)
252	**NGARCH**	1.304	3.088	6.572	12.303	20.144
		(0.013)	(0.018)	(0.019)	(0.016)	(0.010)
	SV	1.301	3.101	6.596	12.316	20.136
		(0.013)	(0.018)	(0.020)	(0.016)	(0.011)

NIG - NGARCH and its diffusion limit based on EGP

NIG-NGARCH (daily frequency) and NIG-SV Prices for European Put Options $S_0 = 100$						
Maturity	**Model**	Strike=80	Strike=90	Strike=100	Strike=110	Strike=120
7	**NIG-NGARCH**	0.000	0.005	1.145	9.979	19.976
		(0.000)	(0.000)	(0.003)	(0.000)	(0.000)
	NIG-SV	0.000	0.002	1.171	9.978	19.976
		(0.000)	(0.000)	(0.003)	(0.000)	(0.000)
21	**NIG-NGARCH**	0.006	0.118	1.977	9.971	19.930
		(0.001)	(0.002)	(0.005)	(0.001)	(0.000)
	NIG-SV	0.003	0.108	2.013	9.956	19.929
		(0.000)	(0.002)	(0.006)	(0.001)	(0.000)
63	**NIG-NGARCH**	0.167	0.792	3.397	10.239	19.811
		(0.004)	(0.007)	(0.009)	(0.004)	(0.001)
	NIG-SV	0.161	0.809	3.431	10.210	19.799
		(0.004)	(0.007)	(0.010)	(0.004)	(0.001)
126	**NIG-NGARCH**	0.636	1.792	4.783	10.920	19.769
		(0.009)	(0.013)	(0.014)	(0.009)	(0.004)
	NIG-SV	0.643	1.817	4.802	10.890	19.733
		(0.010)	(0.014)	(0.015)	(0.009)	(0.003)
252	**NIG-NGARCH**	1.655	3.419	6.773	12.368	20.166
		(0.017)	(0.021)	(0.021)	(0.017)	(0.010)
	NIG-SV	1.663	3.425	6.764	12.329	20.110
		(0.019)	(0.021)	(0.021)	(0.016)	(0.010)

Convergence of Gaussian GARCH option prices to SV

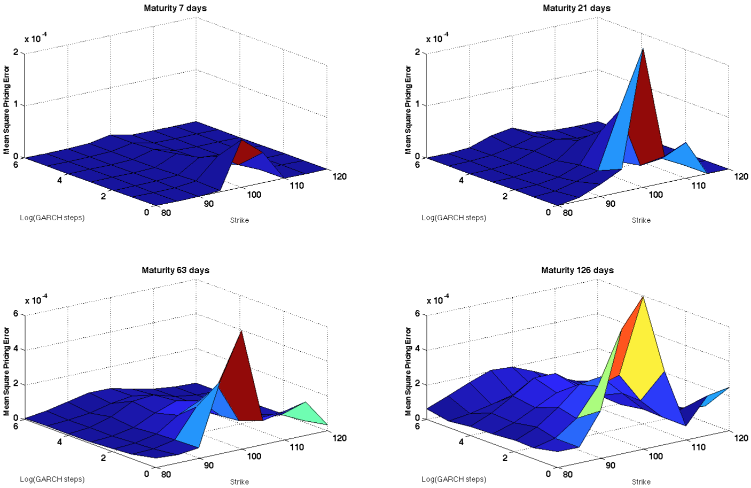

Figure: Convergence of the Gaussian GARCH option prices to their continuous time limit counterparts.

Convergence of NIG-GARCH option prices to SV for EGP

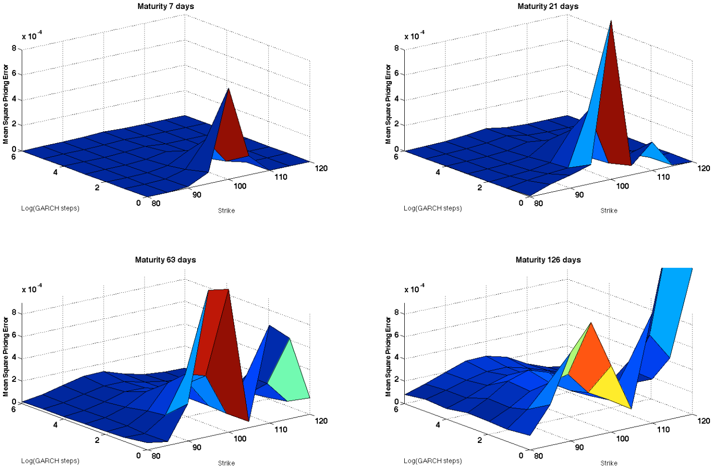

Figure: Convergence of the NIG-GARCH option prices to their continuous time limit counterparts.

Pricing errors for SV models based on ESS and EGP

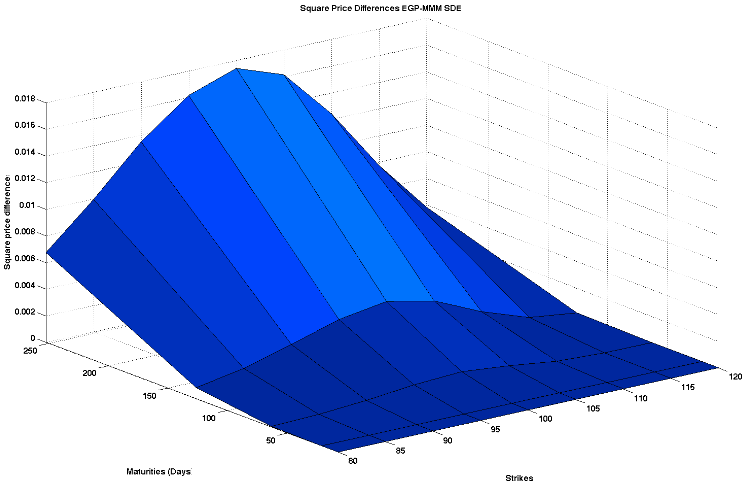

Figure: Square differences between NIG-GARCH diffusion limits prices based and MMM and EGP

NIG - NGARCH and its diffusion limit for ESS

NIG-NGARCH and SV Prices for European Call Options $S_0 = 500$

Maturity	Model	K=450	K=470	K=490	K=510	K=530	K=550
7	NIG-NGARCH	50.116	30.359	12.392	2.034	0.109	0.005
		(0.036)	(0.042)	(0.035)	(0.015)	(0.003)	(0.001)
	SV	50.090	30.289	12.424	2.088	0.064	0.000
		(0.046)	(0.044)	(0.036)	(0.014)	(0.002)	(0.000)
21	NIG-NGARCH	50.880	32.284	16.347	5.714	1.223	0.173
		(0.051)	(0.051)	(0.043)	(0.028)	(0.012)	(0.005)
	SV	50.807	32.261	16.455	5.808	1.154	0.105
		(0.074)	(0.067)	(0.053)	(0.031)	(0.012)	(0.003)
63	NIG-NGARCH	54.903	38.260	23.964	12.987	5.868	2.178
		(0.075)	(0.069)	(0.058)	(0.046)	(0.033)	(0.021)
	SV	54.852	38.267	24.011	13.005	5.790	2.024
		(0.120)	(0.106)	(0.086)	(0.065)	(0.041)	(0.023)
126	NIG-NGARCH	60.865	45.412	31.873	20.743	12.347	6.657
		(0.088)	(0.081)	(0.072)	(0.062)	(0.049)	(0.035)
	SV	60.674	45.225	31.684	20.536	12.108	6.395
		(0.142)	(0.126)	(0.107)	(0.088)	(0.066)	(0.047)
252	NIG-NGARCH	70.954	56.644	43.825	32.721	23.481	16.148
		(0.106)	(0.100)	(0.091)	(0.081)	(0.073)	(0.064)
	SV	70.297	55.952	43.120	32.023	22.812	15.523
		(0.187)	(0.171)	(0.153)	(0.133)	(0.111)	(0.091)

- We investigate the continuous time limit of non-Gaussian GARCH models based on the extended Girsanov principle and the conditional Esscher transform.
- In both cases, the bivariate diffusion limit of the risk-neutralized GARCH process is no longer a standard Hull-White SV model.
- For the Esscher transform case, we recover the weak limit obtained by applying the MMM to the SV model under P.
- For the Extended Girsanov Principle, we recover the MMM only in the case of symmetric distributions. For skewed GARCH innovations, we obtain a non-zero market price of non-hedgeable risk which is proportional to the market price of the equity risk, where the constant of proportionality depends on the skewness and kurtosis of the underlying distribution.
- The numerical results suggest that there are no significant differences between GARCH option pricing models based on a Normal Inverse Gaussian distribution and their SV counterparts.