Modelling Annuity Portfolios and Longevity Risk with Extended CreditRisk$^+$

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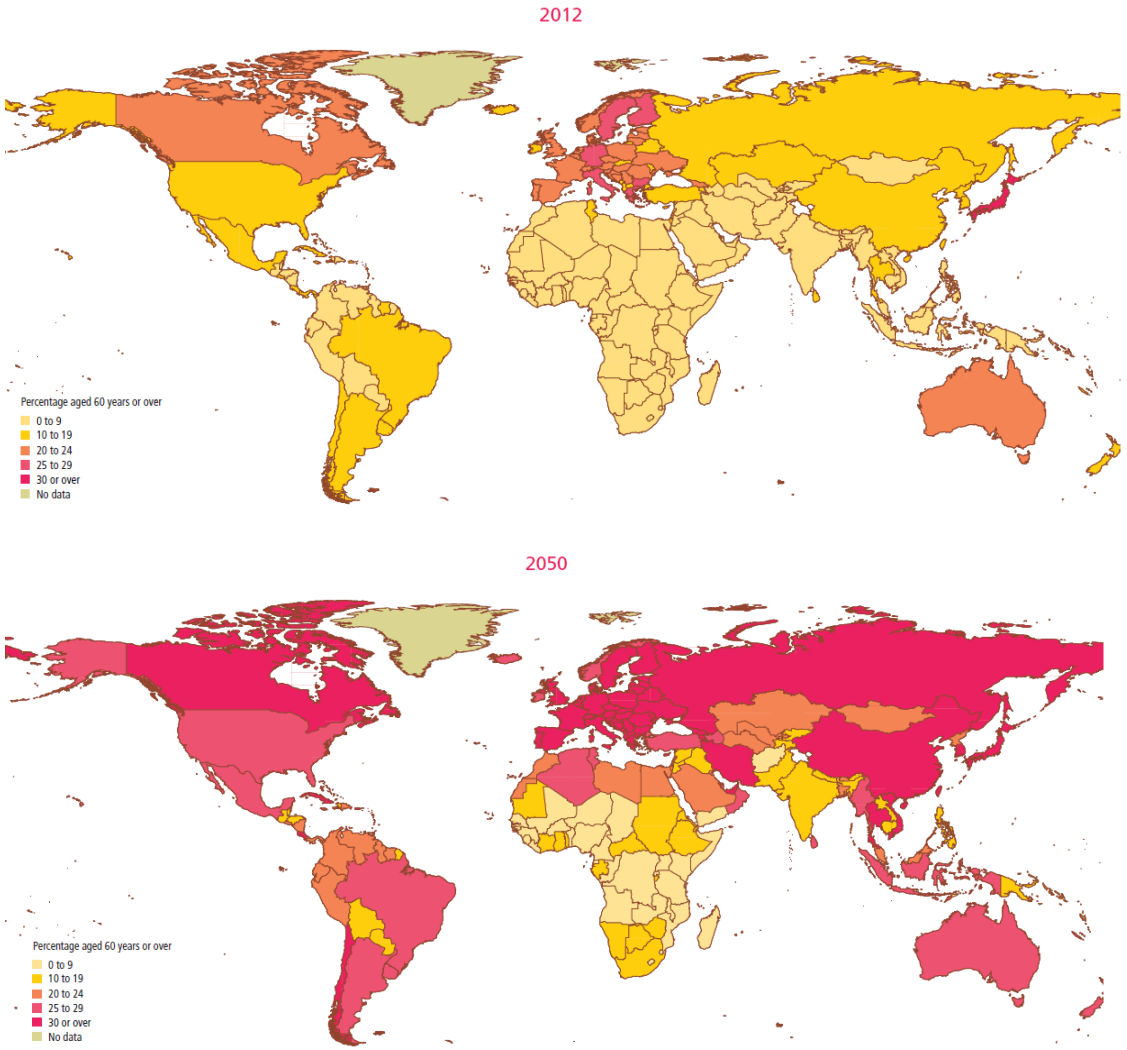
Population Aging - United Nations Data (2012): percentage aged 60 years or over, 2012 vs 2050 forecast



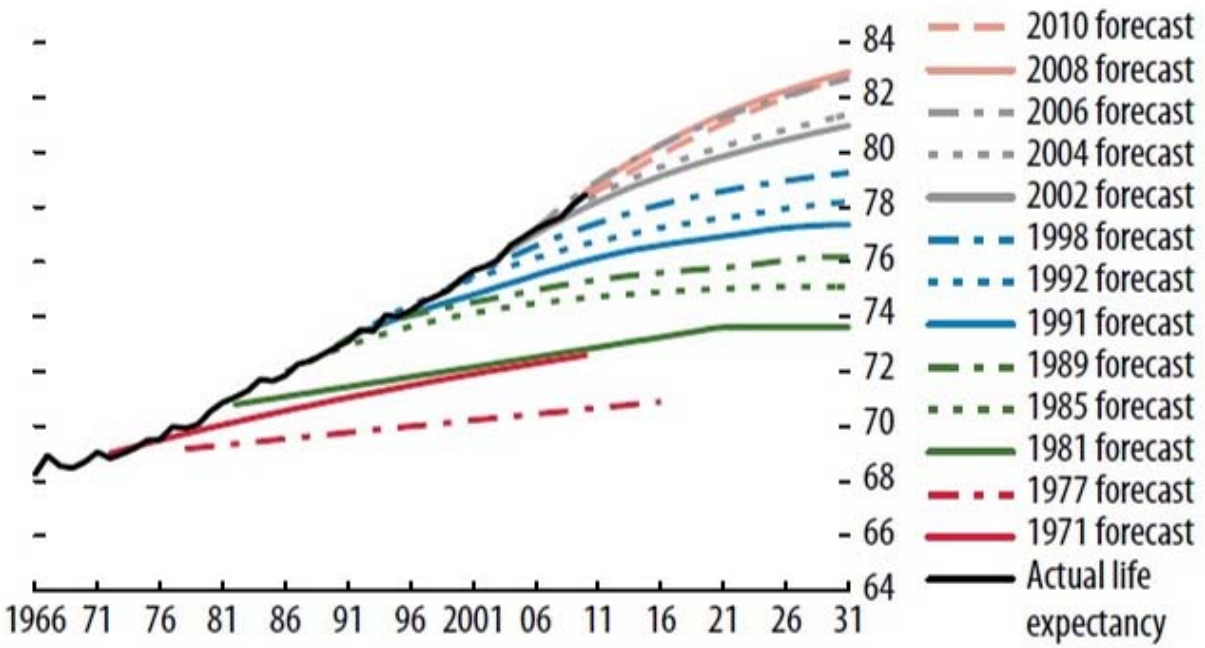
Outline

- We develop a model to derive loss distributions of annuity portfolios over one period.
- The model is based on extended CreditRisk$^+$.
- There exists a numerically stable and fast algorithm to derive loss distributions and risk measures exactly.
- Based on publicly available data we provide estimation procedures, including MCMC.
- The model can also be applied to model life tables and mortality forecasts.
- Stress scenarios can also be tested.
- Setup to model new insurance contracts.

Observation I

When applying life tables to annuities, death probabilities have to be reduced by minimal risk margins to account for longevity (e.g. DAV in Germany):

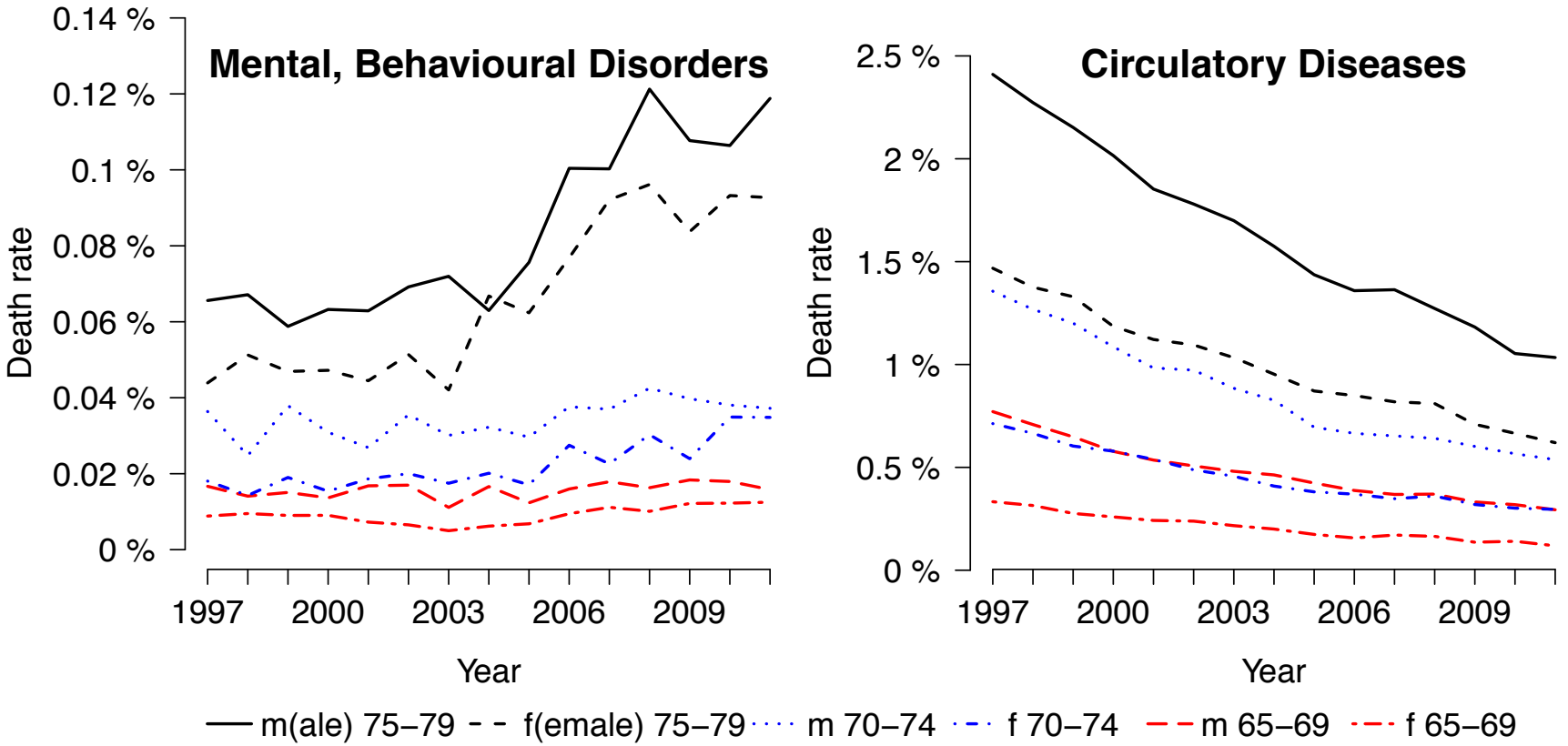
- Mortality trends: For example, Lee–Carter model.
- $\sim 7\%$: Statistical fluctuation.
- 10%: Parameter risk, structural differences.
- 15%: Selection risk.



UK, projected life expectancy at birth for males 1966-2031. Office of National Statistics

Observation II

Australian mortality rates due to different death causes show significant patterns, also on a short-term scale (1997-2011).



Goal

Develop a model which derives **loss distributions** of annuity portfolios over one period which

- takes into account some of the risks mentioned before to account for **longevity**,
- accounts for changes in rates of different **death causes**,
- accounts for dependence between policyholders,
- has a potentially **short runtime** (not Monte Carlo),
- can model **any kind of annuity** (index-linked, variable annuities),
- has the feature of **stress testing**.

Extended CreditRisk$^+$ (ECRP)

Collective risk model *extended CreditRisk*$^+$ (see Schmock (2014), short ECRP, and CRP (1997)) is able to cover all those attributes, if

default is treated as **death** .

We know:

- ECRP allows an explicit calculation of the loss distribution via a stable and fast algorithm.
- ECRP can be applied to any kind of annuity (index-linked, variable annuities).
- ECRP allows flexible handling of dependence through common stochastic risk factors.

Notation and setup

- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and **policyholders** $1, \ldots, m$.
- **Death indicators** $N_1, \ldots, N_m \in \mathbb{N}_0$ (random variables) where $\{N_i = 0\}$ indicates 'no death'.
- Independent stochastic or deterministic **annuity payments** $X_1, \ldots, X_m \in \mathbb{N}_0$ (may be multi-dimensional including discounted actuarial reserve and different lines of business) and annuities which need not be paid in the case of death $Y_1, \ldots, Y_m \in \mathbb{N}_0$, mutually indep. and indep. of $N_1, \ldots, N_m$.

Total portfolio loss

For i.i.d. copies $\{Y_{i,j}\}_{j \in \mathbb{N}}$ of Y_i, for $i \in \{1, \ldots, m\}$, derive

$$L := \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} \sum_{j=1}^{N_i} Y_{i,j} \, .$$

Death indicator

What assumptions on death indicators N_i in

$$S := \sum_{i=1}^{m} \sum_{j=1}^{N_i} Y_{i,j} \, ?$$

- In reality, (N_i) are **Bernoulli** distributed: Monte Carlo.
- If (N_i) are independently **Poisson** distributed with mean (λ_i), then Panjer's recursion can be applied, i.e., for $\lambda := \sum_{i=1}^{m} \lambda_i$, $q_\nu := \sum_{i=1}^{m} (\lambda_i/\lambda) \mathbb{P}(Y_i = \nu)$, and $\mathbb{P}(S = 0) = \exp(-\lambda)$

$$\mathbb{P}(S = s) = \frac{\lambda}{s} \sum_{\nu=1}^{s} \nu \, q_\nu \, \mathbb{P}(S = s - \nu), \quad s \in \mathbb{N}_0.$$

- If (N_i) are **compound Poisson** distributed, then Panjer's recursion can still be applied in some cases as in (extended) CreditRisk$^+$.

Multiple deaths

Multiple deaths of a single policyholder can occur when using (compound) Poisson distributed deaths, but:

Multiple deaths are not a major issue

- Since annual death probabilities are small for most ages, multiple deaths are unlikely.
- Multiple deaths is not a major issue for longevity risk modelling.
- Approximations using Poisson sums are justified by Poisson approximation and generalisations of this result (cf. Vellaisamy and ChaudhuriVellaisamy and Chaudhuri (1996)).
- With proper scaling, we get accurate results (next example).

Illustrative example

- $m = 1\,000$ policyholders with annual death probability $q = 0.05$.
- For policyholder i, let X_i be LogNormal with $\mu = 4$ and $\sigma = 0.5$. Let $U_i \sim U(0, 1]$ and define $Y_i := X_i U_i$.

Using 10 000 simulations derive $S := \sum_{i=1}^{m} \sum_{j=1}^{N_i} Y_{i,j}$ where $Y_{i,j} \sim Y_i$, for N_i being Poisson as well as Bernoulli distributed, both with $\mathbb{P}(N_i = 0) = 1 - q$, $i \in \{1, \ldots, m\}$.

		Bernoulli	Poisson
VaR(S)	0.01	1 007.87	1 005.16
	0.05	1 174.09	1 170.17
	0.15	1 325.84	1 324.91
	0.99	2 333.72	2 373.00

Annuity model using extended CreditRisk$^+$

Annuity model with risk factors

For all policyholders $i \in \{1, \ldots, m\}$:

- **Annual death probability** q_i^* and set $q_i := -\log(1 - q_i^*)$.
- **Risk factors** $\Lambda_1, \ldots, \Lambda_K$ are independent and have gamma distributions with mean 1 and variances $\beta_1, \ldots, \beta_K$.
- Death indicators are split up $N_i = N_{i,0} + N_{i,1} + \cdots + N_{i,K}$ due to different risk factors (**death causes**) with corresponding **weights** $w_{i,0}, \ldots, w_{i,K} \geq 0$ such that $w_{i,0} + \cdots + w_{i,K} = 1$.
- $N_{i,0}$ is independent of everything else, $\mathcal{L}(N_{i,0}) = \mathrm{Poi}(q_i\, w_{i,0})$.
- $(N_{i,k})_{i,k}$ are conditionally independent given $\Lambda_1, \ldots, \Lambda_K$ and they have a **compound Poisson** distribution

$$\mathcal{L}(N_{i,k} \,|\, \Lambda_1, \ldots, \Lambda_K) = \mathcal{L}(N_{i,k} \,|\, \Lambda_k) = \mathrm{Poi}(q_i\, w_{i,k}\, \Lambda_k)$$

Interpretation and comments on the annuity model

- Risk factors $\Lambda_1, \ldots, \Lambda_K$ represent causes of death such as neoplasms, cardiovascular diseases or idiosyncratic components. The variation in this risk factors represents unexpected improvement in medication or outbursts of epidemics, etc.
- E.g., a low value of the risk factor for neoplasms Λ_k reduces the Poisson intensity in $\mathrm{Poi}(q_i\, w_{i,k}\, \Lambda_k)$ and implies reduced death probability which may be the case if a new cancer treatment is available.
- The weights $w_{i,k}$ indicate how vulnerable policyholder i is to risk factor Λ_k.

Algorithm for (extended) CreditRisk$^+$

For $S = \sum_{i=1}^{m} \sum_{j=1}^{N_i} Y_{i,j}$ the algorithm is given by

$$\mathbb{P}(S = \nu) = \frac{\lambda}{\nu} \sum_{n=1}^{\nu} n c_n \mathbb{P}(S = \nu - n), \quad \nu \in \mathbb{N},$$

where $\mathbb{P}(S = 0) = \exp\big(\lambda(c_0 - 1)\big)$ with $\lambda, c_0 \in \mathbb{R}$ and

$$c_\nu = f(b_{1,\nu}, \ldots, b_{K,\nu}), \quad \nu \in \mathbb{N}_0,$$

where, for all $k \in \{1, \ldots, K\}$, $b_{k,0} \in \mathbb{R}$ and

$$b_{k,\nu} = g_\nu(b_{k,1}, \ldots, b_{k,\nu-1}), \quad \nu \in \mathbb{N}_0,$$

with some functions $g_1, g_2, \ldots, f$.

Idea of proof: Deriving the probability-generating function of S for all $z \in \mathbb{C}$ with $\|z\|_\infty \leq 1$ gives

$$\mathbb{E}\big[z^S\big] = \sum_{\nu \in \mathbb{N}_0} \mathbb{P}(S = \nu) z^\nu = \exp\big(\lambda(\tilde{\varphi}(z) - 1)\big), \tag{1}$$

where $\tilde{\varphi}(z) = \sum_{\nu \in \mathbb{N}_0} c_\nu z^\nu$. The form of the probability-generating function implies that S is a Poisson sum, see Schmock (2014).

Available data

- Historical data of annual number of deaths $n_{\mathrm{a,g},k}(t) \in \mathbb{N}_0$ categorised by age $\mathrm{a} \in \{1, \ldots, \mathrm{A}\}$, gender $g \in \{f, m\}$ and death cause $k \in \{0, \ldots, K\}$ for years $t \in \{1, \ldots, T\}$.
- For Australia: Long-term data for 18 age groups, both genders with 19 death causes available.
- Corresponding historical population counts $m_{\mathrm{a,g}}(t)$.

Data and model linkage

$n_{\mathrm{a,g},k}(t)$ correspond to realisations of the random variable

$$N_{\mathrm{a,g},k}(t) := \sum_{i=1}^{m_{\mathrm{a,g}}(t)} N_{i,k}(t)\,,$$

Assumptions for estimation

Simplifying assumptions for consistent estimation

Additionally assume the following:

- Weights and death probabilities are the same within each age category and gender.
- Risk factor variances $\beta_1, \ldots, \beta_K$ are constant over the years.
- All random variables are independent for different points in time.
- **Trends** in death probabilities take the form

$$\log q_{\mathrm{a,g}}(t) = a_{\mathrm{a,g}} + (T - t)\, b_{\mathrm{a,g}}\,,$$

and weights

$$w_{\mathrm{a,g},k}(t) = c_{\mathrm{a,g},k} + (T - t)\, d_{\mathrm{a,g},k}\,.$$

Estimation procedures

Using these assumptions, we can develop several estimation approaches:

- **Matching of moments**: Easy to calculate and reasonably accurate.
- **Maximum a posteriori**: MAP-function is given explicitly but numerical optimisation is impossible ($\sim$400 parameters). Risk factor realisations can be estimated (stress testing) and handy approximations can be derived.
- **Maximum likelihood**: ML-function is given explicitly but numerical optimisation is hard ($\sim$360 parameters).
- **Markov chain Monte Carlo**: Based on likelihood function, switching to a Bayesian setting, parameters can be estimated accurately. This approach is slow but provides the feature of posterior densities of parameters.

Estimation procedures

Lemma (The maximum a posteriori approach)

For $k \in \{1, \ldots, K\}$ and $t \in \{1, \ldots, T\}$, given the posterior density $\pi(\boldsymbol{\beta}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \boldsymbol{\lambda} | \mathbf{n})$ from the maximum a posteriori approach, we get by taking partial derivatives

$$\hat{\lambda}_k(t) = \frac{1/\hat{\beta}_k^{\mathrm{MAP}} - 1 + \sum_{\mathrm{a}=1}^{\mathrm{A}} \sum_{\mathrm{g} \in \{\mathrm{f},\mathrm{m}\}} n_{\mathrm{a},\mathrm{g},k}(t)}{1/\hat{\beta}_k^{\mathrm{MAP}} + \sum_{\mathrm{a}=1}^{\mathrm{A}} \sum_{\mathrm{g} \in \{\mathrm{f},\mathrm{m}\}} \rho_{\mathrm{a},\mathrm{g},k}(t)}$$

as well as

$$\log \hat{\beta}_k^{\mathrm{MAP}} + \frac{\Gamma'\left(1/\hat{\beta}_k^{\mathrm{MAP}}\right)}{\Gamma\left(1/\hat{\beta}_k^{\mathrm{MAP}}\right)} = \frac{1}{T} \sum_{t=1}^{T} \left(1 + \log \hat{\lambda}_k(t) - \hat{\lambda}_k(t)\right),$$

where for given $\hat{\lambda}_k(1), \ldots, \hat{\lambda}_k(T) > 0$, the latter equation has a unique positive solution.

Likelihood function

$$\ell_{\mathbf{n}} = \prod_{t=1}^{T} \left(\left(\prod_{\mathrm{a}=1}^{\mathrm{A}} \prod_{\mathrm{g}\in\{\mathrm{f},\mathrm{m}\}} \frac{e^{-\rho_{\mathrm{a},\mathrm{g},0}(t)}\, \rho_{\mathrm{a},\mathrm{g},0}(t)^{n_{\mathrm{a},\mathrm{g},0}(t)}}{n_{\mathrm{a},\mathrm{g},k}(t)!} \right) \right.$$
$$\times \prod_{k=1}^{K} \left(\frac{\Gamma(1/\beta_k + n_k(t))}{\Gamma(1/\beta_k)\, \beta_k^{1/\beta_k}\, (1/\beta_k + \rho_k(t))^{1/\beta_k + n_k(t)}} \right.$$
$$\left. \left. \times \prod_{\mathrm{a}=1}^{\mathrm{A}} \prod_{\mathrm{g}\in\{\mathrm{f},\mathrm{m}\}} \frac{\rho_{\mathrm{a},\mathrm{g},k}(t)^{n_{\mathrm{a},\mathrm{g},k}(t)}}{n_{\mathrm{a},\mathrm{g},k}(t)!} \right) \right)$$

where, $n_k(t) := \sum_{\mathrm{a}=1}^{\mathrm{A}} \sum_{\mathrm{g}\in\{\mathrm{f},\mathrm{m}\}} n_{\mathrm{a},\mathrm{g},k}(t)$, as well as $\rho_{\mathrm{a},\mathrm{g},k}(t) := m_{\mathrm{a},\mathrm{g}}(t)\, q_{\mathrm{a},\mathrm{g}}(t)\, w_{\mathrm{a},\mathrm{g},k}(t)$ and $\rho_k(t) := \sum_{\mathrm{a}=1}^{\mathrm{A}} \sum_{\mathrm{g}\in\{\mathrm{f},\mathrm{m}\}} \rho_{\mathrm{a},\mathrm{g},k}(t)$.

Illustration example setup

- **Periods** $t \in \{1, \ldots, 10\}$.
- Two **age categories** $(\mathrm{a}_1, \mathrm{a}_2)$ with 10 000 policyholders each and one gender g.
- Annual **death probabilities** between 0.005 and 0.1.
- Two non-idiosyncratic **risk factors** Λ_1, Λ_2 with variances $\beta_1 = 0.05$ and $\beta_2 = 0.2$.
- **Weights** $w_{\mathrm{a}_1,\mathrm{g},1} = 0.1$, $w_{\mathrm{a}_2,\mathrm{g},1} = 0.2$, $w_{\mathrm{a}_1,\mathrm{g},2} = 0.3$ and $w_{\mathrm{a}_2,\mathrm{g},2} = 0.4$ which are assumed to be constant over time.

Number of deaths $n_{\mathrm{a},\mathrm{g},k}(t)$ are then generated via simulated risk factor realisations $(\lambda_1(t), \lambda_2(t))_{t \in \{1,\ldots,10\}}$ and simulation of Poisson distributions with parameters $p_{\mathrm{a}_i,\mathrm{g}}\, w_{\mathrm{a}_i,\mathrm{g},j}\, \lambda_j(t)$.

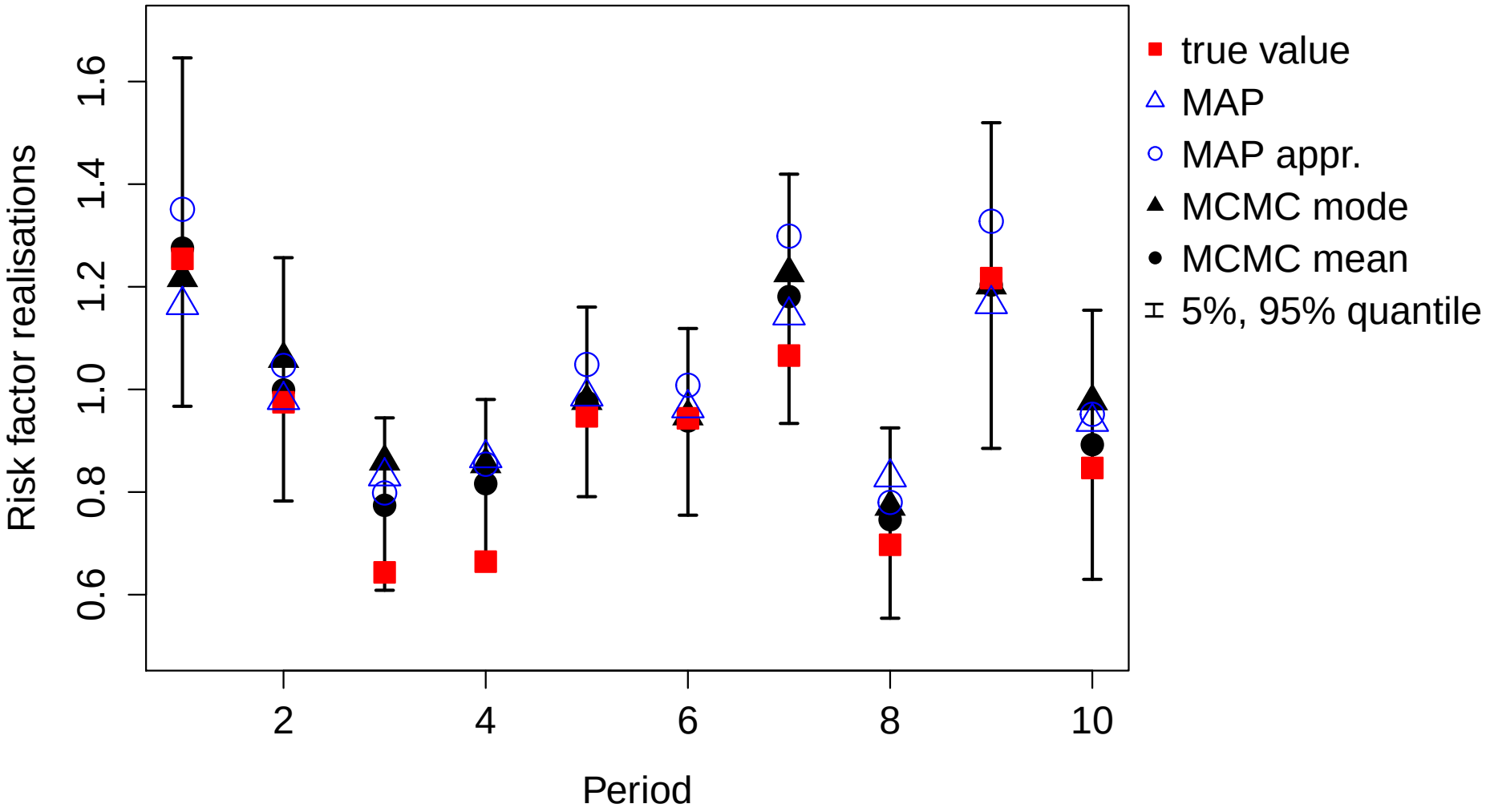
Estimation results

Using all direct estimation procedures as well as MCMC (Random walk Metropolis–Hastings within Gibbs) for the maximum a posteriori approach we get the following:

	β_1	β_2	$c_{a_2,g,1}$	$d_{a_2,g,1}$
true	0.050	0.200	0.200	0.000
MM	0.054	0.267	0.161	0.003
MAP	0.015	0.218	0.158	0.003
MLE	0.032	0.215	0.152	0.006
MAP MCMC single	β_1	β_2	$c_{a_2,g,1}$	$d_{a_2,g,1}$
mode	0.020	0.297	0.148	0.004
mean	0.084	0.377	0.164	0.004
5% quantile	0.025	0.152	0.125	-0.004
95% quantile	0.210	0.793	0.202	0.013
standard error	0.216%	0.441%	0.119%	0.027%

Estimation results: risk factor realisations

Estimates for risk factor realisations and true values.



Real-world example: setup

- Australian death and population data
- **Periods** $t \in \{1997, \ldots, 2011\}$.
- Eight **age categories** 50–54 years,. . . , 80-84 years and 85+ for each gender.
- Ten non-idiosyncratic **risk factors** (death causes) $\Lambda_1, \ldots, \Lambda_{10}$.
- In this setting optimisation over 362 parameters is required.

Using the extended CreditRisk$^+$ setup with log-linear trends for death probabilities and linear trends for weights, we estimate the model via **matching of moments** and **MCMC** (random walk Metropolis–Hastings within Gibbs) with 20 000 steps.

Lee–Carter model vs. our annuity model

Given the number of living people $m_{\mathrm{a,g}}(t)$ as well as annual deaths $n_{\mathrm{a,g}}(t)$, for age a, gender g and years $t \in \{1, \dots, T\}$, the **Lee–Carter model** models death rates $\hat{q}_{\mathrm{a,g}}(t) := n_{\mathrm{a,g}}(t)/m_{\mathrm{a,g}}(t)$ in the form

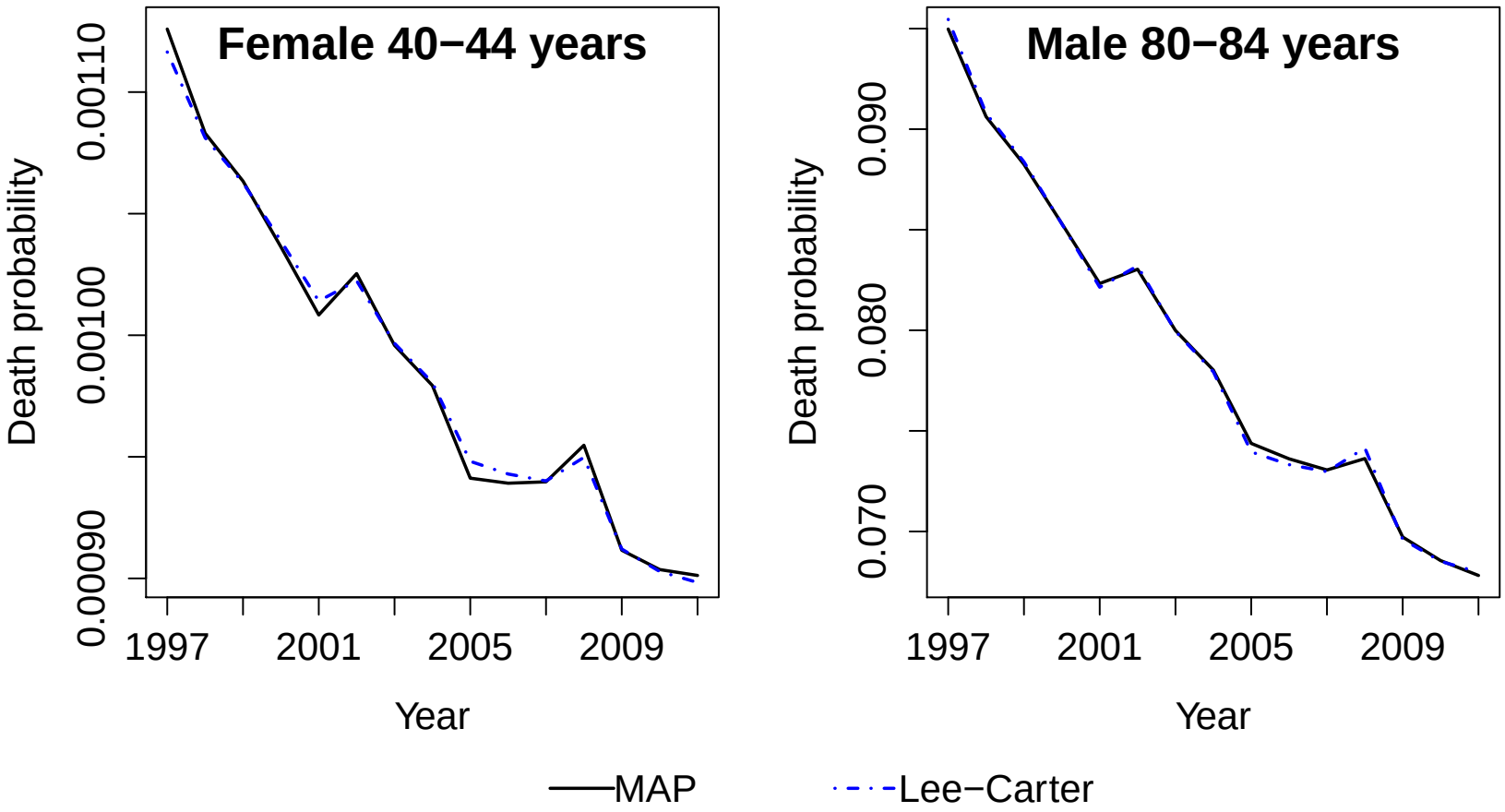
$$\log \hat{q}_{\mathrm{a,g}}(t) = a_{\mathrm{a,g}} + \kappa_t \, b_{\mathrm{a,g}} + \varepsilon_{\mathrm{a,g},t} \,, \quad t \in \{1, \dots, T\} \,,$$

with independent normal error terms $\varepsilon_{\mathrm{a,g},t}$ and mean zero. Using a suitable normalisations, estimates $\hat{a}_{\mathrm{a,g}}$, $\hat{b}_{\mathrm{a,g}}$ and $(\hat{\kappa}_t)_{t \in \{1,\dots,T\}}$ can be derived via method of moments and singular value decompositions.

Lee–Carter model vs. our annuity model

Consider our annuity model with one common risk factor $\Lambda_1(t)$ and weights $w_{\mathrm{a,g},1}(t) = 1$, for all $t \in \{1, \ldots, T\}$. Then, we expect

$$q^{\mathrm{LC}}_{\mathrm{a,g}}(t) \approx q^{\mathrm{MAP}}_{\mathrm{a,g}}(t)\lambda^{\mathrm{MAP}}_1(t), \quad t \in \{1, \ldots, T\}.$$



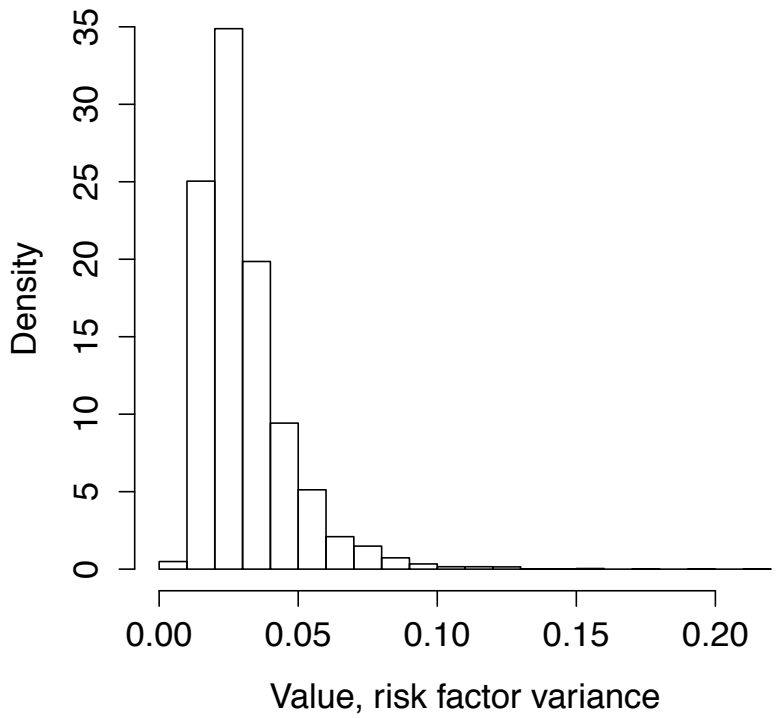
Estimation results: Weights and changes

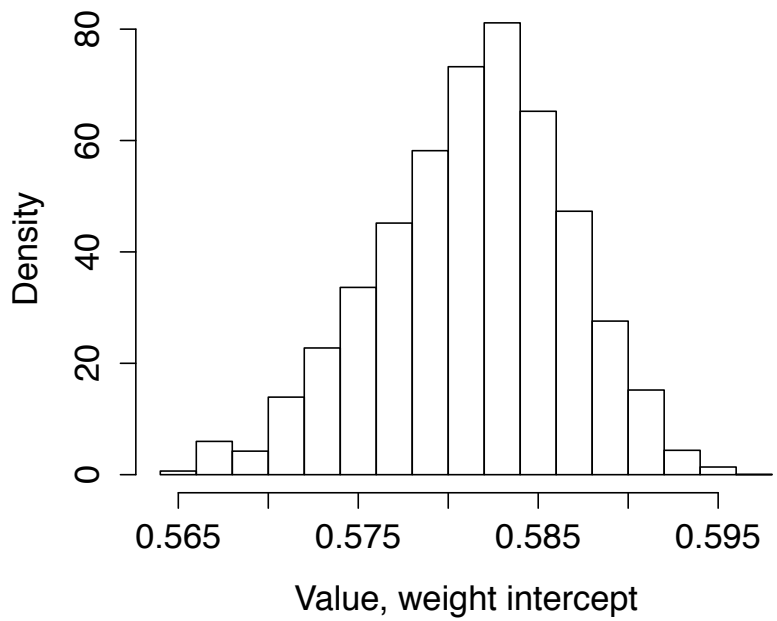
Estimated weights and corresponding trends for different death causes using MCMC posterior mean estimates for males 50–54 years (left) and 80–84 years (right).

	Weight	Shift	Weight	Shift
Infectious & parasitic	2.86%	3.56%	1.38%	0.85%
Neoplasms	36.09%	-0.66%	32.23%	1.45%
Endocrine & nutritional	3.35%	-0.73%	4.64%	2.65%
Mental and behavioural	1.49%	1.72%	4.29%	4.79%
Nervous system	2.83%	1.43%	4.65%	2.66%
Circulatory	23.65%	-1.60%	32.13%	-3.15%
Respiratory system	3.55%	1.18%	10.61%	-0.27%
Digestive system	7.25%	1.90%	2.89%	0.65%
Injury and poisoning	15.84%	1.32%	2.95%	2.70%
Genitourinary system	0.75%	2.38%	2.71%	-0.74%

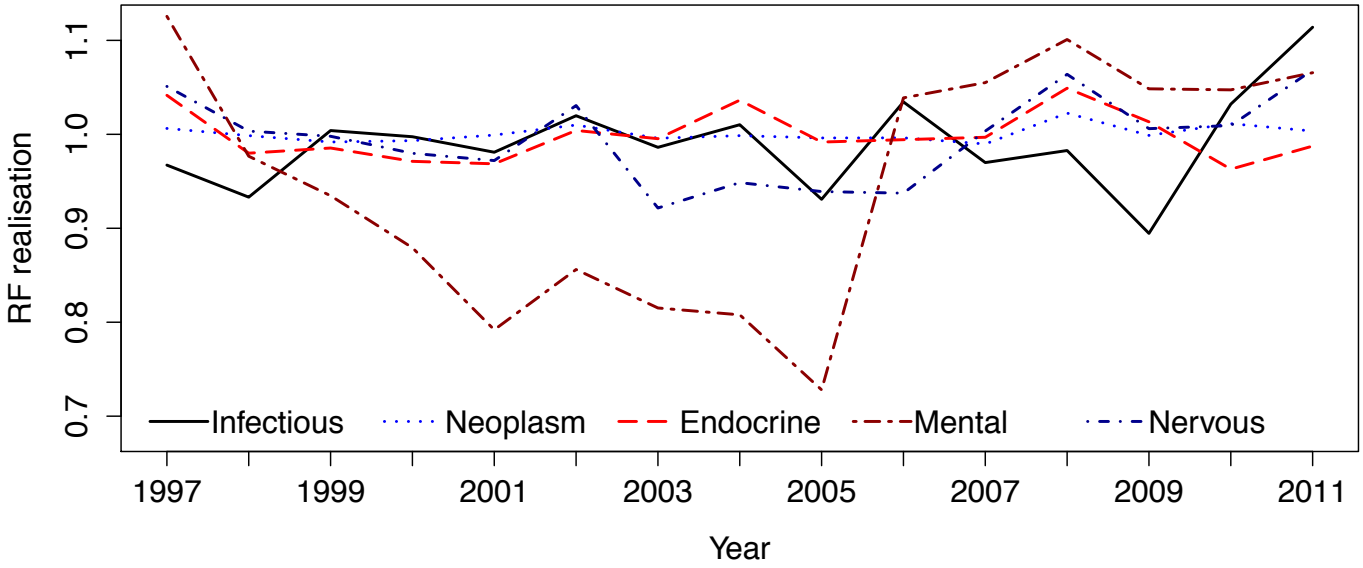
Estimation results: MCMC density histograms

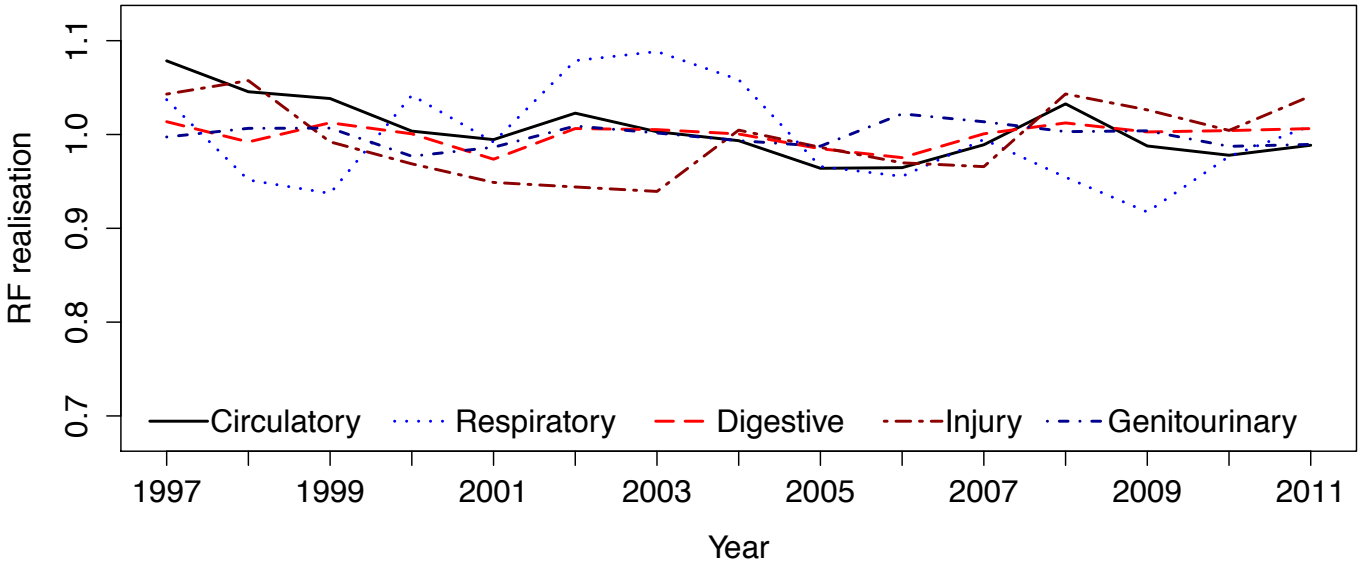
Density histograms of MCMC chains for the variance of risk factor for mental and behavioural disorders as well as for weight intercept for females aged 55 to 59 years of death cause neoplasms (right).





Estimation results: Risk Factors





Forecast: death probabilities and death causes weights

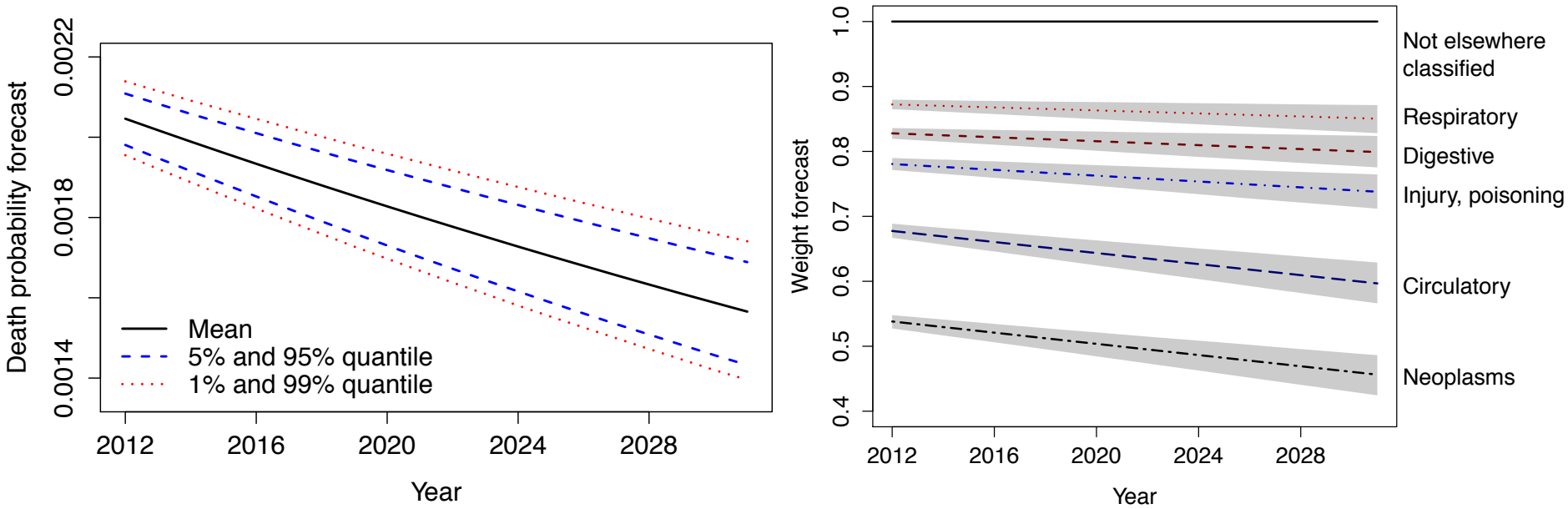
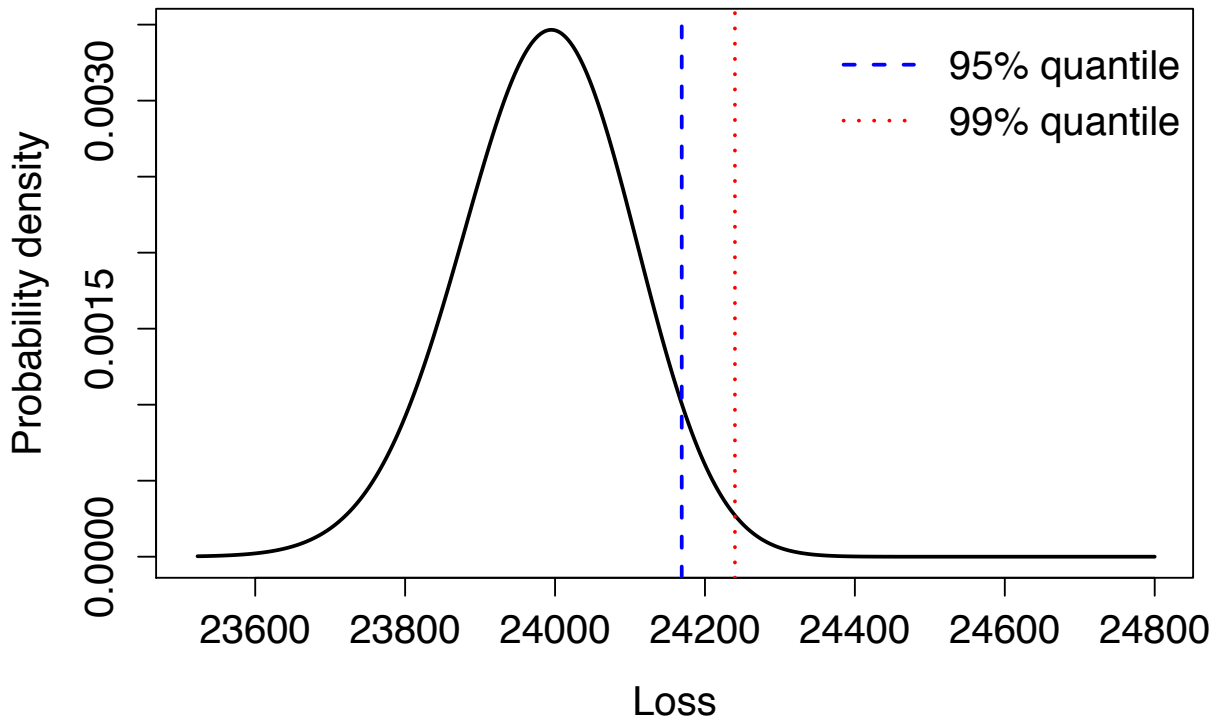


Figure : Forecasted death probabilities and cumulative weights for various death causes for females aged 50-54; shaded area correspond to 5% and 95% quantiles

Parameter uncertainty: portfolio loss distribution

- Australian data with the same setup as before.
- Let each age category and gender have 100 policyholders with annual deterministic annuity payments $X_i = 11, \ldots, 20$.
- Derive loss distribution $L = \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} \sum_{j=1}^{N_i(T+1)} X_{i,j}$ with extended CreditRisk$^+$ where $X_{i,j} \sim X_i$.



Parameter uncertainty: Distribution of quantiles

As we are using MCMC, we can derive (approximatively) distributions of quantiles of L, i.e., we can quantify parameter risk.

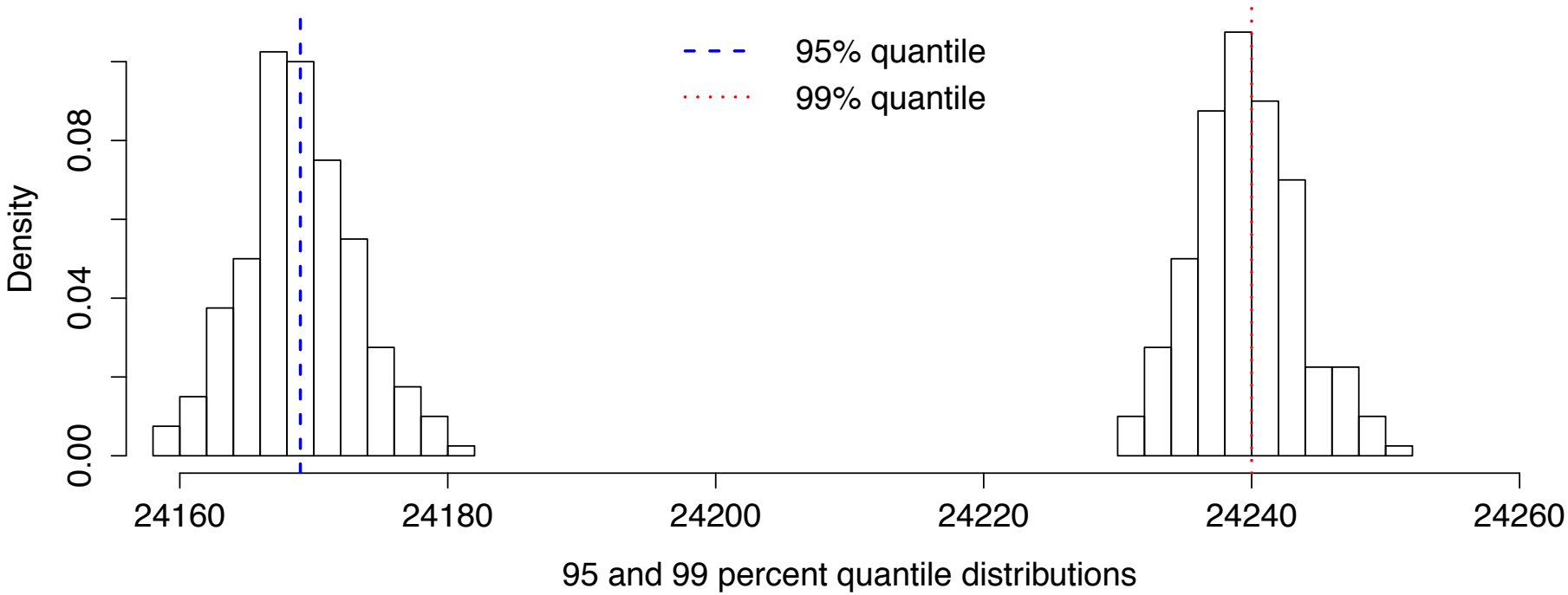
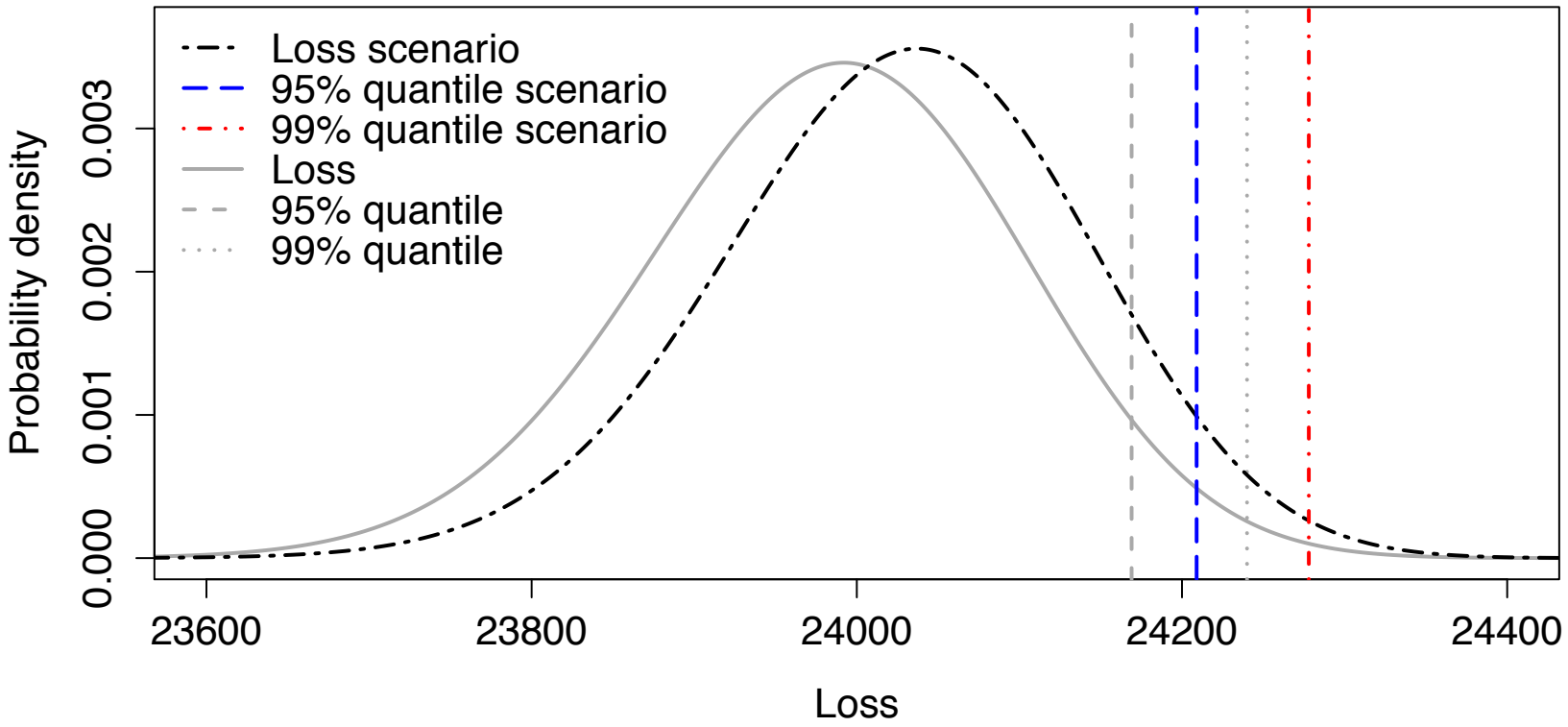


Figure : Distributions of 95 and 99 percent quantile based on MCMC chain realisations.

Scenario analysis

If we assume that deaths due to cancer decrease by 25% over all age categories next year due to better medication, then we get the following shifted distribution of L.



Further applications

- **Population forecasts**.
- Effects of **scenarios** where death rates of certain death causes suddenly spike by $x \cdot 100$ percent within one year can be derived.
- Our model can be generalised to individual losses $Y_{i,k}$ depending on death cause k which allows **modelling of new life-insurance contracts**.
- Certain **dependence structures for risk factors** can be assumed while still being able to calculate loss distributions exactly via iterated Panjer's recursion.

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Thank you for your attention!

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