Free-surface flow under a sluice gate of an inclined wall from deep water

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Abstract. Nonlinear solutions of free surface flow under a sluice gate are studied in this paper. Upstream, the fluid is assumed to be infinitely in depth, and the gate makes an angle β to the horizontal axis. Therefore, the flow emerges from the gate and produces uniform stream far downstream. The problem is solved numerically by a boundary element method derived from an integral equation along the free surface. An analytical function is constructed, relating to the upstream flow, so that the integral equation is solvable. As the result, a free surface flow with smooth detachment from the edge of the gate is obtained for relatively large upstream Froude numbers, otherwise a free surface with back flow near the edge of the gate is indicated, and it tends to a stagnation point for a certain Froude number.

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1. Introduction

We study the steady two-dimensional irrotational flow of an ideal fluid in a domain bounded by an infinite horizontal wall at the bottom, an inclined wall, making angle β to the horizontal axis, presenting a sluice gate; and a free surface as illustrated in Figure 1a. Physically, the infinite depth fluid flows through a slit under the inclined wall, and it forms a stream with free surface as the boundary. Far downstream the stream is uniform. When the net volume flux of the fluid approaching the slit is Q and the width of the slit measured from the bottom wall is D, the free surface profile is observed, especially near the edge of the gate.

Most sluice gate flows are observed for fluid of finite depth in upstream, such as in Frangmeier and Strelkoff [1]; Loroch [2]; Chung [3]; also in Asavanant and Vanden Broeck [4]; Vanden Broeck [5]; and Binder and Vanden Broeck [6]. The solutions are characterized by uniform and supercritical flow far downstream, and the flow far upstream is supercritical or subcritical. A train of waves can be obtained when the upstream flow is subcritical. Binder and Vanden Broeck [7] then developed a problem involving multiple disturbances on the bottom of the channel and the free surface; such as submerged obstacle, pressure distribution and sluice gate. They obtained the solutions with radiation condition, i.e. waves are formed near the gate and disappear, tend to uniform, far upstream. However, all types of solutions have uniform and supercritical flow far downstream. This character is also obtained in this paper, but it is caused from infinite depth fluid in the upstream. The difference with the previous solutions is existing the back flow near the edge of the gate, and it becomes a stagnation point. The free surface separates the inclined wall with angle $2\pi/3$, or $-5\pi/6$ to the wall. This limiting case agrees with the result obtained by Vanden-Broeck and Tuck [8] who observed free surface flow locally near a vertical wall. The problem in this paper is generallization of the vertical sluice gate, solved by by Wiryanto, et. al. [9].

As the method, most of works above solves the problem numerically by boundary element. The method is also used to solve other free surface flows. Wiryanto and Tuck [10,11] applied the method to free surface flow producing one jet and also two jets. For free surface flows caused by a line sink or source, we can read for example in Wiryanto [12]; and Hocking and Forbes [13,14]. The boundary element method is constructed from an integral equation of hodograph complex variable corresponding to particle velocity. In expressing the real part of the variable into the imaginary one, Cauchy integral theorem is applied. The problems using the boundary element method usually have the hodograph variable Ω satisfying conditions of Cauchy integral theorem, i.e. analytic and $\Omega(\zeta) \to 0$ as $|\zeta| \to \infty$. ζ is artificial complex variable as the result of conformal mapping of physical plane. However, in this study we involve infinite depth of the fluid. This causes the second condition of Cauchy integral theorem not satisfied anymore, since the fluid velocity far upstream is uniform radially. Therefore, we need to construct the appropriate function for Cauchy integral theorem.

The construction of the analytic and bounded function is explained in Section 2. The similar problem has been done for the case of zero gravity by Wiryanto [15], but the horizontal wall is terminated so that the flow becomes a waterfall, and analytical solutions are obtained. In section 3, the numerical procedure in solving the integral equation is presented. The integration is approximated by trapezoidal method involving unknown variables. A system of nonlinear algebraic equation is then constructed from the integral equation, and it is solved by Newton iteration method. As a result, we present in section 4 some plots of the surface profile and discus the numerical observation.

2 Formulation

We consider the steady two-dimensional irrotational flow of an inviscid and incompressible fluid in a dam of infinite depth, bounded by an inclined wall as a sluice gate with width of slit D. We choose Cartesian coordinates with the x-axis along the bottom and y-axis directed vertically upward intersecting the edge of the inclined wall. The net volume of the flux in the dam is Q per unit distance perpendicular to the plane of flow, and the flow is assumed to leave the edge of the gate tangentially, see Figure 1a.

From the assumption of the fluid and the flow, we present the stream in a complex potential $f = \phi + i\psi$ corresponding to the complex velocity $df/dz = u - iv$, where $z = x + iy$. For convenience, we work in non-dimensional variables by taking Q as the unit flux and D as the unit length, and we define $\phi = 0$, $\psi = 0$ at the center coordinates of

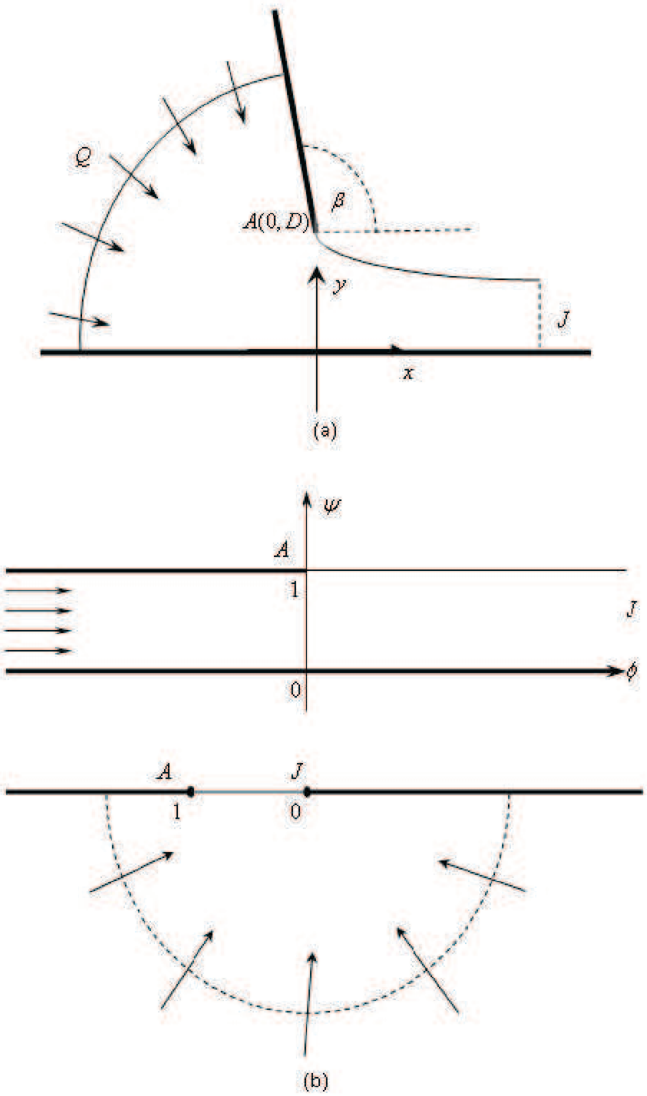


Figure 1: Sketch of the flow under a sluice gate (a) in physical z-plane, (b) in f-plane and artificial ζ-plane.

physical plane z. Therefore, the f-plane is a strip with width 1, also the non-dimensional width of the slit. Now, our task is to solve the boundary value problem

$$\nabla^2\phi = 0$$

in the flow domain. The dynamic condition is expressed by Bernoulli equation

$$\frac{1}{2}F^2\left(\phi_x^2 + \phi_y^2\right) + y = \text{constant} \tag{1}$$

along the free boundary representing hydrostatics pressure. F is Froude number defined as

$$F = \frac{Q}{\sqrt{gD^3}}, \tag{2}$$

where g is acceleration of gravity. The other condition is kinematic along the solid and free boundaries, satisfying

$$\frac{\partial\phi}{\partial\bar{n}} = 0 \tag{3}$$

where $\bar{n}$ is a normal vector of the boundaries. Physically, this condition represents that fluid particles on the boundary remain on it.

In determining ϕ, we first introduce a hodograph variable $\Omega = \tau - i\theta$ having relationship to the velocity vector

$$\frac{df}{dz} = e^{\Omega}. \tag{4}$$

Meanwhile, the flow domain in f-plane is mapped into a half lower artificial plane $\zeta = \xi + i\eta$ by

$$f = -\frac{1}{\pi}\log\zeta. \tag{5}$$

The downstream is mapped to $\zeta = 0$ and the edge of the gate A is mapped to $\zeta = -1$. The schematic diagram of the flow is shown in Figure 1b. The bold lines correspond to the solid boundary, and the thick line corresponds to the free boundary.

Instead of determining ϕ, we solve the hodograph variable Ω with respect to the artificial variable ζ, satisfying

$$\nabla^2\Omega = 0$$

subject to (the dynamic condition (1) becomes)

$$\frac{1}{2}F^2e^{2\tau} + y = c, \quad -1 < \xi < 0 \tag{6}$$

where c is an unknown constant; and the kinematic condition (3) becomes

$$\theta = \begin{cases} \beta - \pi, & -\infty < \xi < -1 \\ 0, & 0 < \xi < \infty \end{cases} \tag{7}$$

Here θ is unknown for $-1 < \xi < 0$.

Relation between θ and τ is then required in reducing the unknown variables. This can be obtained by using Cauchy integral theorem. As the complex function, we define

$$\chi(\zeta) = \Omega + \frac{\beta - \pi}{\pi}\log\zeta \tag{8}$$

This function is analytic and $\chi \to 0$ for $|\zeta| \to \infty$, so that it can be applied to Cauchy integral theorem. We come up to (8) since the upstream flow far from the slit is uniform with velocity $df/dz \to 0$ and the streamlines bouncing by horizontal and vertical walls having angle θ as given in (7). Logarithm function is the one having character described above, so that

$$\Omega \to -\frac{\beta - \pi}{\pi} \log \zeta, \quad \text{for } |\zeta| \to \infty$$

and this is used to construct χ as written in (8).

In applying χ to Cauchy integral theorem along closed path covering the flow domain in ζ-plane, it is enough to consider along the real ξ-axis giving

$$\chi(\xi) = -\frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\chi(s)}{s - \xi} ds \tag{9}$$

The function χ is then expressed in τ and θ for both sides in (9), and the real part gives

$$\tau(\xi) = -\frac{\beta - \pi}{\pi} \log |1 + \xi| + \frac{1}{\pi} \mathrm{PV} \int_{-1}^{0} \frac{\theta}{s - \xi} ds \tag{10}$$

for $-1 < \xi < 0$. Here, PV is used to denote Cauchy principal-value for the integration.

The last part, which has to be prepared before obtaining the integral equation, is to determine y along the free surface. We use (4) and (5) to have

$$\frac{dz}{d\zeta} = -\frac{e^{-\Omega}}{\pi\zeta} \tag{11}$$

and the imaginer part along the free surface gives

$$\frac{dy}{d\xi} = -\frac{e^{-\tau} \sin\theta}{\pi\xi} \tag{12}$$

Therefore, the value y is obtained by integrating (12) giving

$$y(\xi) = 1 - \int_{-1}^{\xi} \frac{e^{-\tau(s)}}{\pi s} \sin\theta(s) ds \tag{13}$$

τ in (13) is evaluated from (10). The formulae y in (13) and τ in (10) are then substituted to (6) giving the integral equation which has to be solved.

3. Numerical Procedure

The nonlinear integral equation (6) converts to a set of N algebraic equations in N unknowns, if we approximate the integration (10) by summation in a suitable manner. The interval of integration (0,1) is first discretized by defining the end-points of $N - 1$ subintervals $\xi_0 = -1 < \xi_1 < \xi_2 \cdots < \xi_{N-1} = -\epsilon$, and then we let $\theta_j = \theta(\xi_j)$ for $j = 1, 2, \cdots, N - 1$, be $N - 1$ unknowns. $-\epsilon$ is a small value representing the position of free surface relatively far from the slit, and we need this number to truncate the integration (10), as it is impossible to know the end of the free surface, but we need the rest subinterval $(-\epsilon, 0)$.

In order to evaluate the Cauchy principle-value singular integral in (10), we approximate $\theta(\xi)$ as varying linearly on the interval (ξ_{j-1}, ξ_j), and evaluate the integral over each such interval exactly. For any $\xi_j^* \in (\xi_{j-1}, \xi_j)$, $\tau(\xi_j^*)$ is evaluated by

$$\begin{aligned}\tau(\xi_j^*) \approx & -\frac{\beta-\pi}{\pi}\log|1+\xi_j^*| \\ & + \sum_{l=1}^{N-1}(\theta_{l-1}-\theta_l) + \left\{\theta_l + (\theta_{l-1}-\theta_l)\frac{\xi_j^*-\xi_l}{\xi_{l-1}-\xi_l}\right\}\log\left|\frac{\xi_{l-1}-\xi_j^*}{\xi_l-\xi_j^*}\right|\end{aligned} \tag{14}$$

Similarly, the integral (13) determining the y-coordinate of the free surface can be evaluated by numerical approximation, such as trapezoidal rule

$$\begin{aligned}y(\xi_j^*) & \approx y(\xi_{j-1}^*) \\ & -\frac{1}{2}\left(\frac{e^{-\tau(\xi_j^*)}}{\pi\xi_j^*}\sin\theta(\xi_j^*) + \frac{e^{-\tau(\xi_{j-1}^*)}}{\pi\xi_{j-1}^*}\sin\theta(\xi_{j-1}^*)\right)(\xi_j^*-\xi_{j-1}^*)\end{aligned} \tag{15}$$

In obtaining the N algebraic equations, we use N collocation points ξ_j^* as the mid-point in each subinterval (ξ_{j-1}, ξ_j), except $\xi_N = -\epsilon/2$ and also $\theta(\xi_j^*)$ defined linearly between θ_{j-1} and θ_j. For each point ξ_j^*, the integral equation (6) gives one algebraic equation, so that there are N equations for unknowns $\theta_1, \theta_2, \cdots, \theta_{N-1}$ and the constant c in (6). The parameter Froude number F is given, also define $\theta_0 = -\pi/2$ at the edge of the gate. This closed form is then solved numerically by Newton method. When the iteration converges, N-point coordinates (x_j, y_j) of the free surface are determined from

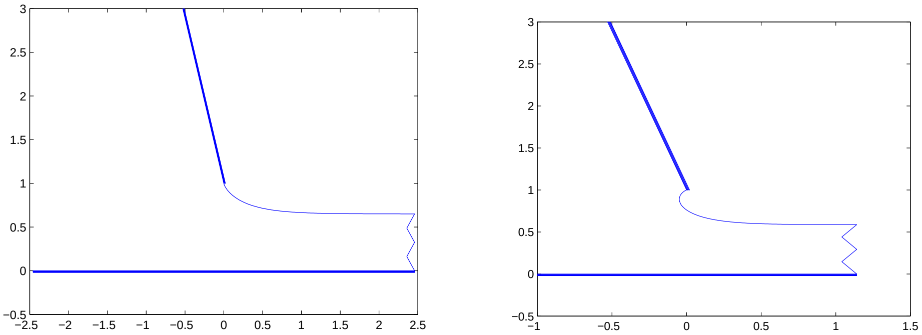
$$\frac{dx}{d\xi} = -\frac{e^{-\tau}}{\pi\xi}\cos\theta \tag{16}$$

and (15) for y. Numerical integration is applied to (16) to get $x(\xi^*)$, using θ obtained in the previous process. We then plot the coordinates (x_j, y_j) to get the surface profile.

4. Results

Most calculations of the numerical procedure described above use $N = 250$ and $\epsilon = 0.000001$. Typical free surface for moderate Froude number is shown in Figure 2. The flow produces stream with smooth free surface leaving the vertical wall, no wave on the free surface; and the stream tends to uniform far downstream, the fluid depth is less than the width of the slit. We computed the result in Figure 2 for $F = 0.8$ and $\beta = 5\pi/9$. For higher Froude numbers, we obtain stream with slightly deeper fluid.

For small Froude numbers F, we are interested in observing the values θ near the edge of the gate. We plot θ versus ξ for $F = 0.8$, corresponding to Figure 2a, and some other $F = 0.4$, 0.3, and 0.25, shown in Figure 3. For $F = 0.8$, we obtain an increasing θ-curve by increasing ξ, and $\theta > \beta - \pi = -1.396$, meaning the direction of the stream separating the wall, similarly the θ-curve for $F = 0.4$ but lower than the curve for $F = 0.8$. Our calculations can be continued for smaller F, and we obtain some values of θ is less than -1.396, i.e. for $F < 0.37$. We record the minimum value of the curve by denoting the coordinates θ_{min} at ξ_{min}, and we collect from calculation using various



(a) (a)

Figure 2: (a) Free surface flow under a sluice gate for $F = 0.8$, $\beta = 5\pi/9$; (b) Free surface with a stagnation point.

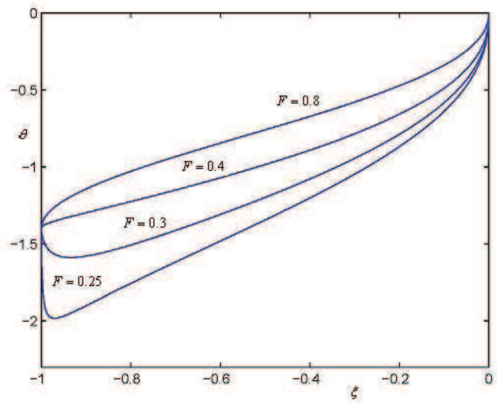


Figure 3: Plot θ versus ξ, for $F = 0.8$ (top), 0.4, 0.3, 0.25 (bottom).

F. The coordinates are then plotted as shown in Figure 4, indicating that the stream separates the wall smoothly, but it turns back, deeper in the left side of the separation point by decreasing θ_{min}, for smaller Froude number. The position of θ_{min} is also shifted, following the curve in Figure 4, it returns back to the separation point, i.e. $\xi_{min} \to -1$. Our numerical scheme is fail for $F < 0.24$ by indicating a sharp change of the θ-curve near the separation point, since we define $\theta(-1) = \beta - \pi$ conflicting to the value at the next discrete point ξ. Therefore, we extrapolate F for $\theta_{min} \to \beta - 4\pi/3$, giving $F = 0.225$, as the limiting case, characterized by 120 degrees of the stream angle, see Vanden-Broeck and Tuck [8]. The free surface flow has a stagnation point at the edge of the inclined wall. We show this limiting free surface flow in Figure 2b.

Now, we perform the result of our calculation when the wall is inclined with the angle $\beta = 0.38\pi$. Far upstream the flow has Froude number $F = 0.25$, and it emerges under the wall forming a free surface flow without wave, as shown in Figure 5a. Similar to the previous result, the stream separates the wall smoothly. Free surface with back flow is

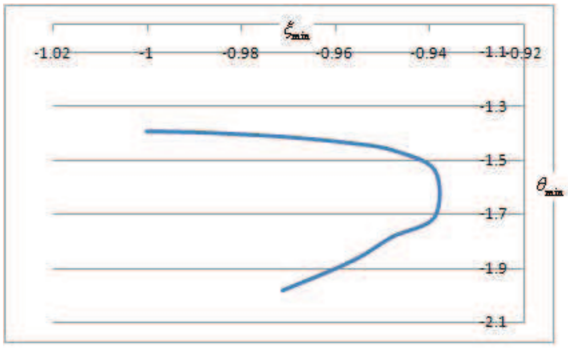
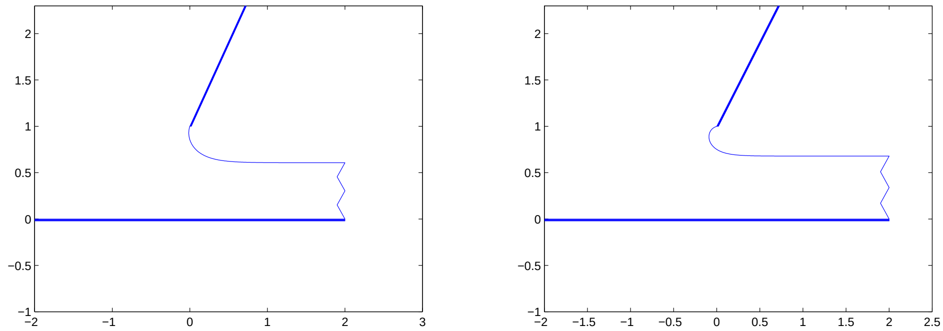


Figure 4: Plot of θ_{min} versus ξ_{min}.



(a) (a)

Figure 5: (a) Free surface flow for $\beta = 0.38\pi$, $F = 0.25$, (b) Free surface with a stagnation point.

indicated obtained for $F < 0.25$, and a stagnation is obtained for $F = 0.13$, shown in Figure 5b. Smaller β, the free surface flow with a stagnation point can be obtained for smaller F, and it should be until $\beta = \pi/3$. Smaller than that angle, it is impossible to get the stagnation point, since for that number the stream separates the wall horizontally to make the inner angle 120 degree to the wetted wall, otherwise the stream goes up, against the gravity.

Conclusions

We have solved numerically the free surface flow under a sluice gate from deep fluid by boundary element method. The deep fluid at the upstream requires an analytical complex function constructed not directly from the hodograph variable, but it includes a term representing the character far from the slit of the gate. As a result, the free surface flow without waves, but smooth detachment at the edge of the wall inclined with angle β for relatively large Froude numbers. Meanwhile solutions with back flow occur for small Froude numbers, and its limiting flow has a stagnation point at the edge of the gate. All types of solutions are uniform and supercritical ($F_\infty > 1$) far downstream.

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