Enclosing and Existence of Cycle Systems

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Graph Decompositions

• Partition the edges of your favorite graph so that each element of the partition induces something interesting.  

$K_7 =$

Use colors on the edges to denote the partition.

Maybe you like paths, but with variety!
Existence Problems

- When do 4-cycle systems exist (of $K_n$)?
- Clearly the number of edges must be divisible by 4
- And the degree of each vertex must be even.
- So $n$ must be congruent to 1 modulo 8.

Each edge has an associated "difference".
Another 4-cycle system: $K_{17}$

Sometimes we need something more complicated

Rotate left to right

Pure Difference 4

Mixed Difference 4

What’s missing??

Mixed differences 2 and -2
Other Cycle Lengths

• The existence of $m$-cycle systems of order $n$ has been solved after a long history.

• Clearly
  – the number of edges must be divisible by $m$
  – the degree of each vertex must be even, and
  – we need $n$ to be at least $m$, or $n = 1$.

Alspach, Gavlas, Šajna (Hoffman, Lindner, Rodger)
Embeddings

- A 4-cycle system $P$ of $\lambda K_v$ is said to be embedded in a 4-cycle system $Q$ of $\lambda K_{v+u}$ if $P$ is a sub-multiset of $Q$.

$P$ with $\lambda = 1$  $Q$ has $\lambda = 1$ too!
Embeddings - History

• The Lindner problem of embedding a partial 3-cycle system of order n into an STS(v) has been solved! (Bryant and Horsley)
  • A necessary condition requires that \( v \geq 2n+1 \)
• For 4-cycles the situation is messier, but recent progress has been dramatic:
  • Necessarily \( v \geq n+n^{1/2}-1 \)
  • Lindner had the best result of \( 2n+15 \) until recently:
    • \( n + 12^{1/2}n^{3/4} + o(n^{3/4}) \) (Lindner and Hilton)
    • \( n + n^{1/2} + o(n^{1/2}) \) (Füredi and Lehel)
Embeddings - History

- Partial 3-cycle system of order $n$ into an STS($v$)
  - $v \geq 2n+1$  
    (Bryant, Horsley)
- Partial 4-cycles systems:
  - $n + n^{1/2} + o(n^{1/2})$  
    (Füredi, Lehel)
- Partial 5-cycle systems:
  - $(9n + 146)/4$  
    (Martin, McCourt)
- Partial $2k$-cycle systems
  - Around $kn$  
    (Hoffman, Lindner, Rodger)
- Partial $2k+1$-cycle systems
  - Around $(4k+2)n$  
    (Lindner, Rodger, Stinson)
Enclosings

- A $k$-cycle system $P$ of $\lambda K_v$ is said to be enclosed in a $k$-cycle system $Q$ of $(\lambda+\mu)K_{v+u}$ if $P$ is a sub-multiset of $Q$. [Diagram]

$P$ with $\lambda = 1$  $Q$ with $\lambda+\mu = 2$

$k = 5$
Conjecture

A 5-cycle system of $\lambda K_v$ can be enclosed in a 5-cycle system of $(\lambda+\mu)K_{v+u}$ if and only if

1. $(\lambda+\mu)(v+u-1)$ is even,
2. The number of new edges is divisible by 5,
3. If $u = 1$ then $\mu(v-1) \geq 3(\lambda + \mu)$,
4. If $u = 2$ then
   \[ \mu v(v-1)/2 - 2(\lambda + \mu) - (v-1)(\lambda + \mu)/2 \geq 0, \]
   and
5. If $u \geq 3$ then
   \[ \mu v(v-1)/2 + (\lambda + \mu)u(u-1)/2 \geq vu(\lambda + \mu)/4 + 2\varepsilon \]

where $\varepsilon = 0$ or 1 if $vu(\lambda + \mu)$ is 0 or 2 (mod 4) resp.

(Asplund, Keranen and Rodger)
These Conditions are Necessary

- Suppose \( u = 1 \).
- The number of 5-cycles including the added vertex, \( \infty \), must be \( v(\lambda + \mu)/2 \).
- Each of these uses 3 edges in \( K_v \).
- So \( \mu v(v-1)/2 \geq 3v(\lambda + \mu)/2 \).
These Conditions are Necessary

• Suppose \( u = 2 \).
• The number of 5-cycles joining the two added vertices must be \( (\lambda + \mu) \).
• Each of these uses exactly 2 edges in \( K_v \).
These Conditions are Necessary

• Suppose \( u = 2 \).

• The number of remaining edges joining the 2 new vertices to \( K_v \) is \( 2v(\lambda+\mu) - 2(\lambda+\mu) \).

• Each of the 5-cycles using these \( 2(v-1)(\lambda+\mu) \) edges uses at least 1 edge in \( K_v \).
These Conditions are Necessary

- Suppose $u = 2$.
- So
  \[ 2(\lambda + \mu) + \frac{2(v-1)(\lambda + \mu)}{4} \leq \frac{\mu v(v-1)}{2} \]
- $(v-1)(\lambda + \mu)$ is even

$v$ vertices

$v_1 \quad v_2$
Theorem

A 5-cycle system of $\lambda K_v$ can be enclosed in a 5-cycle system of $(\lambda+\mu)K_{v+1}$ if and only if

1. $(\lambda+\mu)(v+u-1)$ is even,
2. The number of new edges is divisible by 5, and
3. $\mu(v-1) \geq 3(\lambda + \mu)$.

(Asplund, Keranen and Rodger)
An idea of the proof

Good news!
The number of edges that occur in 5-cycles completely contained in the $\mu K_v$ is exactly

$$\mu v(v-1)/2 - 3v(\lambda + \mu)/2 = \alpha v$$

so is a multiple of $v$. ($\alpha$ is always an integer)

$$(3(\lambda + \mu)/2)v$$ edges occur in 3-paths.

We are in with a chance of using difference methods!
An example will suffice!

\( v = 50, \mu = 1 \)

So the necessary condition

\[ \mu(v-1) \geq 3(\lambda + \mu) \]

means that

\[ \lambda \leq \mu(v-4)/3 = 15.3 \]

so

\[ \lambda \leq 14. \]

We start with the small values and work our way up.
Skolem Sequences

The two integers, $k$, appear $k$ apart in:

\[ 1 \ 1 \ 3 \ 4 \ 2 \ 3 \ 2 \ 4 \]
\[ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \]

These can be represented by pairs:

\{1,2\} \ {3,6} \ {4,8} \ {5,7}\}

Or we can add a constant to each number in each pair:

\{4,5\} \ {6,9} \ {7,11} \ {8,10\}
\( v = 50 = 10k, \mu = 1, \lambda = 2: \text{ so } \alpha = 20 \)

\{4,5\} \{8,10\} \{6,9\} \{7,11\}

Differences used:
\[4,5,8,10,6,9,7,11 = 2k+1\]
\[16, 17, 18, 19\]
\[12, 13, 14, 15\]
\[23, 22, 21, 20 = 4k\]

What is left??
\[1, 2, 3, 24, 25\]
What do we do with differences $1, 2, 3, 24, 25$?

These edges occur in 5-cycles with $\infty$.

Rotate this through 50 positions: $\lambda = 2$.

Rotate this through 25 positions: $\mu = 1$. 
\[ \nu = 50, \mu = 1, \]
What happens if \( \lambda \) is bigger?

- \( \lambda \) must be even
- \( \alpha = \mu(v-1)/2 - 3(\lambda + \mu)/2 \)
- So increasing \( \lambda \) by 2 means that \( \alpha \) is decreased by 3
$v = 50, \mu = 1, \lambda = 4$: so $\alpha = 17$

How do we drop to 17 differences on the right??

Look at the purple 5-cycle: 8, 10, 13, 17, 20
\[ v = 50, \mu = 1, \lambda = 4: \text{ so } \alpha = 17 \]

How do we drop to 17 differences on the right??

Look at the purple 5-cycle: 8, 10, 13, 17, 20
\( v = 50, \mu = 1, \lambda = 4: \) so \( \alpha = 17 \)

The purple 5-cycle pieces: 8, 13, 17

Two more purple 5-cycle pieces: 10, 20
What happens when \( \lambda = 6; \) so \( \alpha = 14 \)?

- When \( \alpha = 14 \), somehow we need to use edges of 4 differences and partition them into 5-cycles!
- We can use edges of difference 10 and 20, like before, but we don’t have 2 more options.
- The edges of differences 1, 2, and 3 can be partitioned into sets that induce 5-cycles!
Remember the Skolem Sequences??

The two integers $k$ appear $k$ apart in:

1 1 3 4 2 3 2 4
1 2 3 4 5 6 7 8

These can be represented by pairs:

{1,2} {3,6} {4,8} {5,7}

Or we can add a constant to each number in each pair:  Why add 3??

{4,5} {6,9} {7,11} {8,10}

So we avoid using differences 1,2 and 3 in the 5-cycles in $K_{50}$!
Notice that only edges of differences 1, 2 and 3 are used.

Here is the second of three base cycles.

Each of these will have multiples of 5 added to them.
Notice that only edges of differences 1, 2 and 3 are used.

Look at the edges of difference 1 in these cycles.

Each of these will have multiples of 5 added to them.
Notice that only edges of differences 1, 2 and 3 are used.

Look at the edges of difference 1 in these cycles.

Each of these will have multiples of 5 added to them.
Notice that only edges of differences 1, 2 and 3 are used.

Look at the edges of difference 1 in these cycles.

Each of these will have multiples of 5 added to them.

The 5 edges of each difference start at vertices that are different mod 5.
\( v = 50, \mu = 1, \lambda = 6: \text{ so } \alpha = 14 \)

{4,5} {8,10} {6,9} {7,11}

Differences used:
4,5,12,16,23
6,9,14,18,21
10,
1,2,3

What is left??

Group these in 3’s:
8,15,17,20
7,11,13,19,22
24,25 gives \( \mu = 1 \)
What happens when $\mu$ increases?

When $v = 50$ and $\mu = 1$, $\lambda \leq 14$.

When $\mu = 2$, the necessary condition

$$\mu(v-1) \geq 3(\lambda + \mu)$$

means that

$$\lambda \leq \frac{\mu(v-4)}{3} = 30.6$$

so

$$\lambda \leq 30.$$  

So every case can be handled using the $\mu = 1$ result  

EXCEPT WHEN $\lambda = 30!!$
Is this $v=50$ case typical?

- For the most part, yes.
- The smallest value of $\lambda$ is also a problem:

<table>
<thead>
<tr>
<th>$\mu$: $v$ (mod 10)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>8</td>
<td>2</td>
<td>6</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>2</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>
Theorem (in all likelihood!)

A 5-cycle system of $\lambda K_v$ can be enclosed in a 5-cycle system of $(\lambda+\mu)K_{v+2}$ if and only if

1. $(\lambda+\mu)(v+u-1)$ is even,

2. The number of new edges is divisible by 5, and

3. If $u = 2$ then
   \[ \mu v(v-1)/2 - 2(\lambda + \mu) - (v-1)(\lambda + \mu)/2 \geq 0 \]

(Asplund, Keranen and Rodger)
Conjecture

A 5-cycle system of $\lambda K_v$ can be enclosed in a 5-cycle system of $(\lambda + \mu)K_{v+u}$ with $u \geq 3$ if and only if

1. $(\lambda + \mu)(v+u-1)$ is even,
2. The number of new edges is divisible by 5, and
3. $\mu v(v-1)/2 + (\lambda + \mu)u(u-1)/2 \geq vu(\lambda + \mu)/4 + 2\varepsilon$

where $\varepsilon = 0$ or 1 if $vu(\lambda + \mu)$ is 0 or 2 (mod 4) respectively.

(Asplund, Keranen and Rodger)
There is a gap!

\[ \mu v(v-1)/2 + (\lambda + \mu)u(u-1)/2 \geq vu(\lambda + \mu)/4 + 2\varepsilon \]

is quadratic in \( v \).

So as \( v \) increases with the other 3 parameters held constant, enclosings may become impossible for some interval, then become possible again.

For example when \( \mu = 1, \lambda = 34 \) and \( u = 7 \), * requires:
\[ v \leq 10 \text{ or } v \geq 120 \]
Finally – he’s in a photo!
Ian Roberts!
Theorem

A 4-cycle system of $\lambda K_v$ can be enclosed in a 4-cycle system of $(\lambda+\mu)K_{v+u}$ if and only if

1. $(v+u-1)(\lambda+\mu)$ is even,
2. The number of new edges is divisible by 4,
3. If $u = 1$ then $\mu(v-1)/2 \geq \lambda + \mu$, and
4. If $u = 2$ then $\mu v(v-1)/2 \geq \lambda + \mu$.

(Newman and Rodger)
When $u \geq 3$ it is not hard to settle.

Use existing results on maximum partial 4-cycle systems.

Add at least 3 vertices

It is easy to use the edges that join 2 new vertices!

And it’s easy to use the other types of edges independently.
Settling $u = 2$ has a graph theoretic feel

- Solving this case involves:
  - Equitable partial 4-cycle systems,
  - Directed Euler Tours, and
  - Expanding nearly-regular graphs into copies of $K_{2,2}$. 

Partial Cycle Systems

• A set of edge disjoint 4-cycles in $K_n$ is said to be a *partial* 4-cycle system of order $n$.

This is a partial 4-cycle system of order 11 that is EQUITABLE.
Equitable Partial Cycle Systems

• **Equitable**: for each pair of vertices $u$ and $v$, the number of cycles containing $u$ differs by at most one from the number of cycles containing $v$.

These are VERY useful!

• 3-cycles: Andersen, Hilton and Mendelsohn
• 4-cycles and 5-cycles: Raines and Staniszló
• Any mixture of cycle lengths! Bryant, Horsley and Maenhaut

• **When can you partition the edges of $K_n$ into two equitable partial cycle systems of two given lengths (say 3 and 5)***?
Recall: If $u = 2$ then $\mu v(v-1)/2 \geq \lambda + \mu$

Add $u = 2$ vertices.

Eventually $\mu$ more edges are used between each pair of vertices.

So

$$\mu v(v-1)/2 \geq (\lambda + \mu)$$

There are $\lambda + \mu$ edges joining the 2 added vertices.
 Sufficiency with $u = 2$: $\mu v(v-1)/2 \geq \lambda + \mu$

So exactly $\mu v(v-1)/2 - (\lambda + \mu)$ edges “must” be in 4-cycles joining vertices in $P$.

In $P$, $\mu$ edges between each pair of vertices need to be used in 4-cycles.

We just saw: there are $\lambda + \mu$ edges joining the 2 added vertices, each of which must be in 4-cycles like this.
Sufficiency with $u = 2$: $\mu v(v-1)/2 \geq \lambda + \mu$

So start with an *equitable* partial 4-cycle system $C_1$ of $\mu K_v$ with exactly $\mu v(v-1)/2 - (\lambda + \mu)$ edges!

It turns out that this number is divisible by 4.

And it is not negative!
Sufficiency with $u = 2$: $\mu v(v-1)/2 \geq \lambda + \mu$

Now look at the complement in $\mu K_v$ of $C_1$. It has exactly $\lambda + \mu$ edges!

All vertices have even degree, so form a directed Euler tour.

In $P$, $\mu$ edges between each pair of vertices ARE NOW used in 4-cycles.

Let $C_2$ be the set of these 4-cycles.
Sufficiency with $u = 2$: $\mu v(v-1)/2 \geq \lambda + \mu$

The remaining edges induce a bipartite graph $B$ from $P$ to $\{S, E\}$.
Since $C_1$ is equitable, each vertex $v$ in $P$ has degree $2s$ or $2s+2$ in $B$ (for some $s$).
Half of the edges incident with $v$ join it to $S$, half join it to $E$.

Form a graph on $V(P)$ in which each vertex $v$ has degree $d_B(v)/2$.
For each edge add a 4-cycle.
Other Enclosings

For 3-cycle systems:

• The problem remains open.
• There are earlier results by Colbourn and Hamm, and also with Rosa (South-Eastern Conference in 1985).
• There are several recent results by Hurd, Munson and Sarvate that consider small enclosings.
• One of the necessary conditions is quadratic.
• Enclosings do not exist in the interval:
\[(v+1)(1-(1-(4mv)/(v-1)^2(\lambda+m)^2)^{1/2}, (v+1)(1 + (1-(4mv)/(v-1)^2(\lambda+m)^2)^{1/2}\]
• Recent result approach this gap from both sides (Newman and Rodger

Nothing appears to be known for longer cycles.
Spouse Avoiding Dinners

Try to find a way for 4 couples to sit at 2 tables, each seating 4 people so that each sits next to each other person exactly once.

Not the spouses!
Spouse Avoiding Dinners

Try to find a way for 4 couples to sit at 2 tables, each seating 4 people so that each sits next to each other person exactly once.

Not the spouses!

Can you do this so that each table has 2 men and 2 women?
Must one avoid one’s spouse??

No! You now have an excuse for another dinner!

Cycle systems of graphs other than $K_n$ are also interesting.

Join vertices in the same group with $\lambda_1$ edges and vertices in different groups with $\lambda_2$ edges

$\lambda_1 = 2$ and $\lambda_2 = 1$

Pure and Mixed Edges
Cycle Systems with 2 Associate Classes

Maybe you have one big table!

There exists a $C_{ap}$-factorization of $K(a,p;\lambda_1,\lambda_2)$ if and only if:

1. $\lambda_1(a-1) + \lambda_2a(p-1)$ is even, and
2. $\lambda_2a(p-1) \geq \lambda_1$.

(Bahmanian, Rodger)
Cycle Systems with 2 Associate Classes

Tables of size 4 are more common!

Suppose $a$ is even.

There exists a $C_4$-factorization of $K(a,p;\lambda_1,\lambda_2)$ if and only if
1. $4$ divides $ap$
2. $\lambda_1$ is even, and
3. If $a \equiv 2 \pmod{4}$ then $\lambda_2 a (p-1) \geq \lambda_1$.

(Billington, Rodger)
Cycle Systems with 2 Associate Classes

Tables of size 4 are more common!

Suppose $a \equiv 1 \pmod{4}$.

There exists a $C_4$-factorization of $K(a, p; \lambda_1, \lambda_2)$ if and only if
1. 4 divides $p$
2. $\lambda_2 > 0$ and is even, and
3. $\lambda_2 a(p-1) \geq \lambda_1$,

except possibly if $a = 9$ and $\lambda_1$ is odd.

(Rodger, Tiemeyer)
What about $a \equiv 3 \pmod{4}$?

Looks difficult from my point of view!
Why must $\lambda_2 a(p-1) \geq \lambda_1$?

Suppose $a \equiv 2 \pmod{4}$.

Consider one $C_4$-factor.

Every part must contain at least 2 vertices incident with mixed edges.

So each $C_4$-factor must contain at least $p$ mixed edges!

The same argument works for hamilton cycles.

$K(a,p;\lambda_1,\lambda_2)$
How do the proofs go?

You need a different perspective!

For the hamilton cycles, use amalgamations!

That approach also lets you prove embedding results!
4-cycle systems of $K(a,p;\lambda_1,\lambda_2)$

There exists a 4-cycle system of $K(a,p;\lambda_1,\lambda_2)$ if and only if
1. Each vertex has even degree,
2. The number of edges is divisible by 4,
3. If $a = 2$ then
   • $\lambda_2 > 0$, and
   • $\lambda_1 \leq 2(p-1) \lambda_2$
4. If $a = 3$ then
   • $\lambda_2 > 0$, and
   • $\lambda_1 \leq 3(p-1) \lambda_2/2$ \hspace{2cm} if $\lambda_2$ is even, and
   • $\lambda_1 \leq 3(p-1) \lambda_2/2 - (p-1)/9$ \hspace{2cm} if $\lambda_2$ is odd.

(Hung Lin Fu, Rodger)
For 3-cycles: Fu, Rodger, Sarvate
For block designs: Bose and Shimamoto – 1952!
Why is $\lambda_1 \leq 3(p-1) \frac{\lambda_2}{2}$ when $a = 3$?

- Every 4-cycle must use at least 2 mixed edges.
- So $3p\lambda_1 \leq 9\lambda_2 p(p-1)/2$
Is $\lambda_1 \leq 3(p-1) \frac{\lambda_2}{2} - \frac{(p-1)}{9}$ when $\lambda_2$ is odd?

- There are an odd number of edges between each pair of parts!
- So some 4-cycles must use at least 3 mixed edges
- So $3p\lambda_1 - \frac{p(p-1)}{6} \leq 9\lambda_2\frac{p(p-1)}{2} - \frac{p(p-1)}{2}$

$K(a, p; \lambda_1, \lambda_2)$

Each of these uses an even number of mixed edges.
Plenty More!

• Cycle systems that cover 2-paths (4-cycles)  
  (Heinrich and Nonay, Cox and Rodger)

• Resolvable versions  
  (Kobayashi and Nakamura)

• Fair and gregarious cycle systems of multipartite graphs

• Cycle systems of line graphs of $K_n$ and of line graphs of complete multipartite graphs (4-cycles)  
  (Rodger and Sehgal)

• Cycle systems (3- and 4-cycles) of $K_n$ minus any graph with  
  – maximum degree 3  
  – One vertex of any degree, all others of degree at most 2  
  (Fu, Fu and Rodger, Sehgal, Ash)
Thanks for listening!

Not quite time for a cuppa!!