Enclosing and Existence of Cycle Systems

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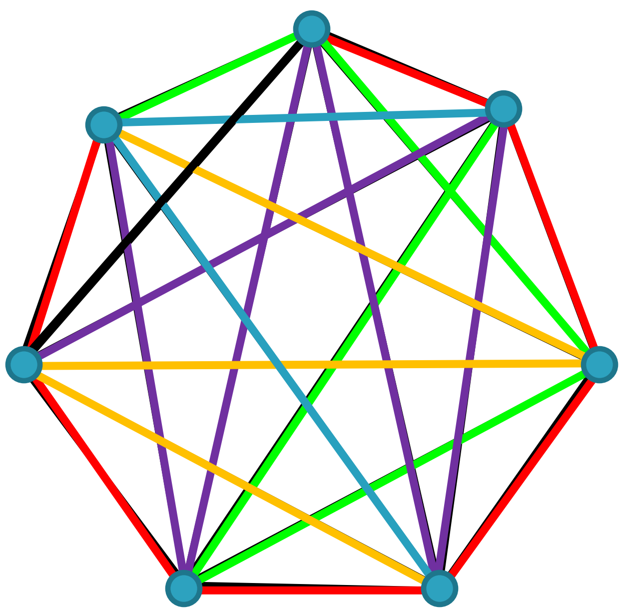
Graph Decompositions

- Partition the edges of your favorite graph so that each element of the partition induces something interesting.

Use colors on the edges to denote the partition.

$K_7 =$

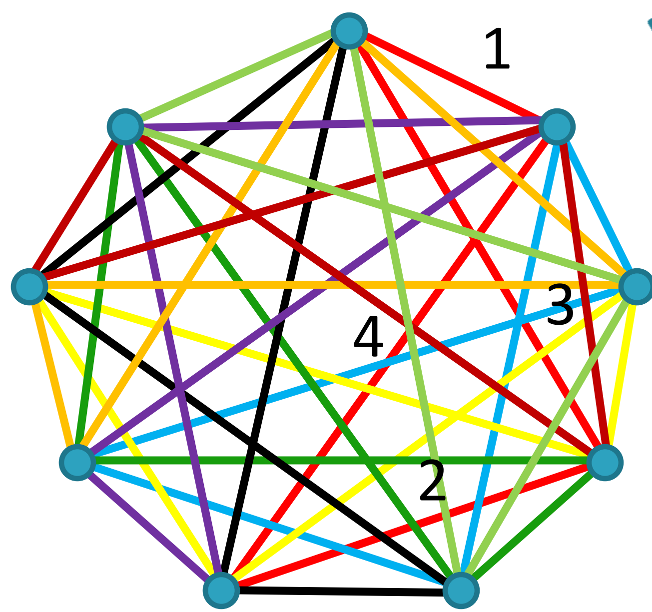


Maybe you like paths, but with variety!

Existence Problems

- When do 4-cycle systems exist (of K_n)?
- Clearly the number of edges must be divisible by 4
- And the degree of each vertex must be even.
- So n must be congruent to 1 modulo 8.

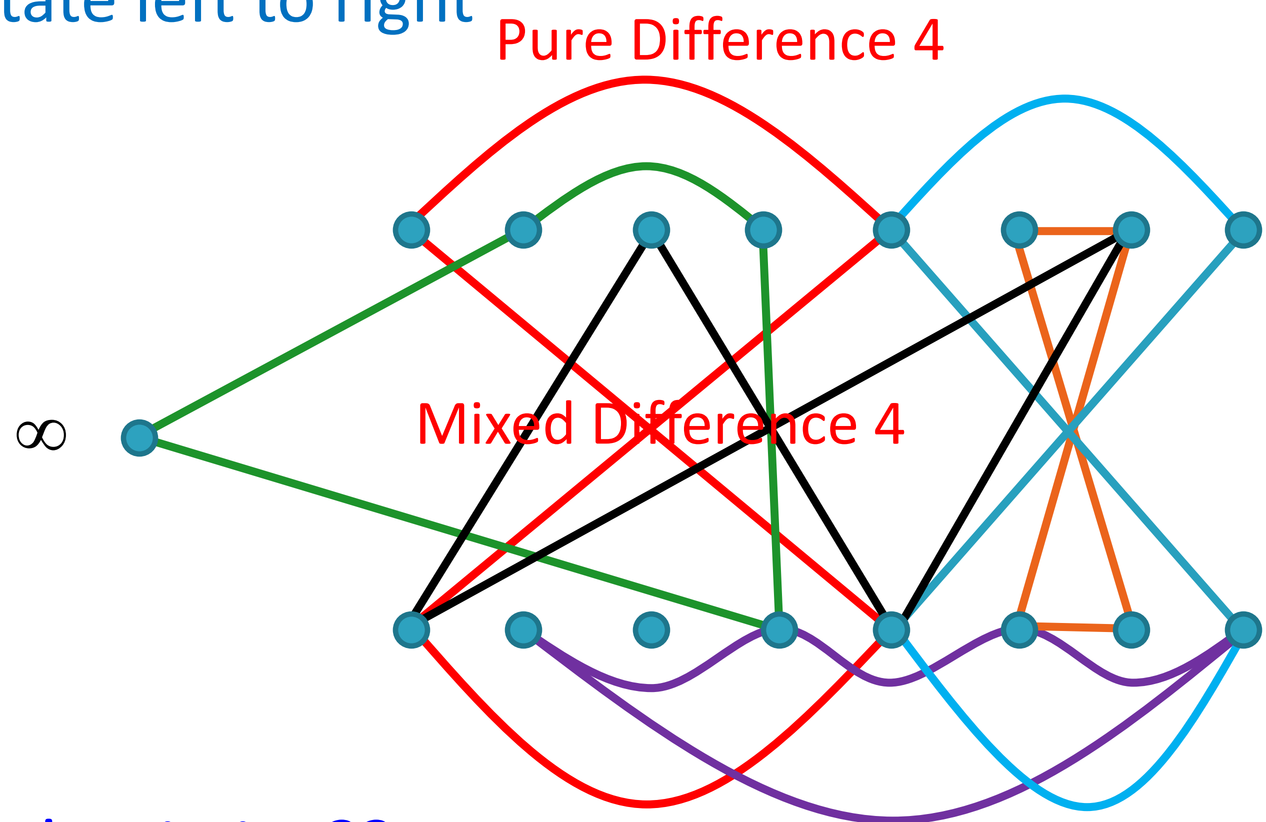
Each edge has an associated "difference".



Another 4-cycle system: K_{17}

Sometimes we need something more complicated

Rotate left to right



What's missing??

Mixed differences 2 and -2

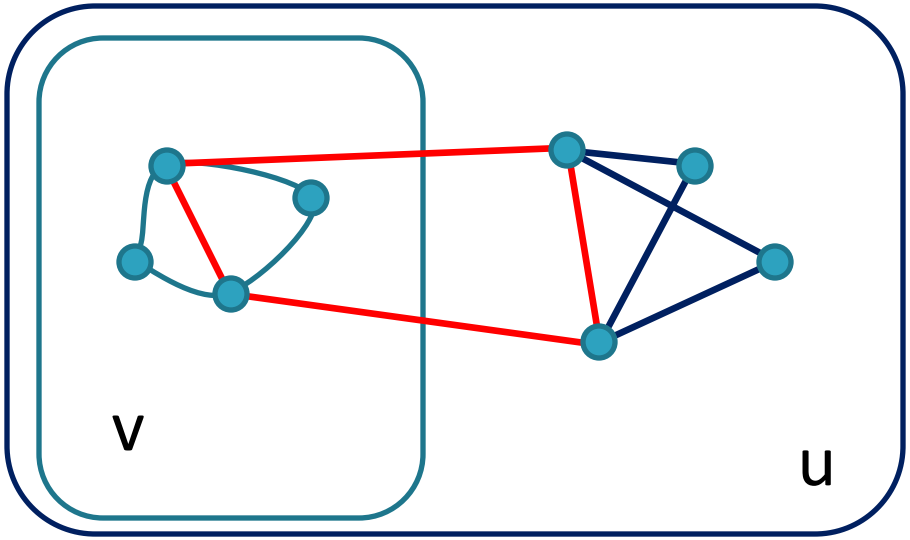
Other Cycle Lengths

- The existence of m-cycle systems of order n has been solved after a long history.
- Clearly
  - the number of edges must be divisible by m
  - the degree of each vertex must be even, and
  - we need n to be at least m, or $n = 1$.

  Alspach, Gavlas, Šajna (Hoffman, Lindner, Rodger)

Embeddings

- A 4-cycle system P of λK_v is said to be embedded in a 4-cycle system Q of λK_{v+u} if P is a sub-multiset of Q.



P with $\lambda = 1$ Q has $\lambda = 1$ too!

Embeddings - History

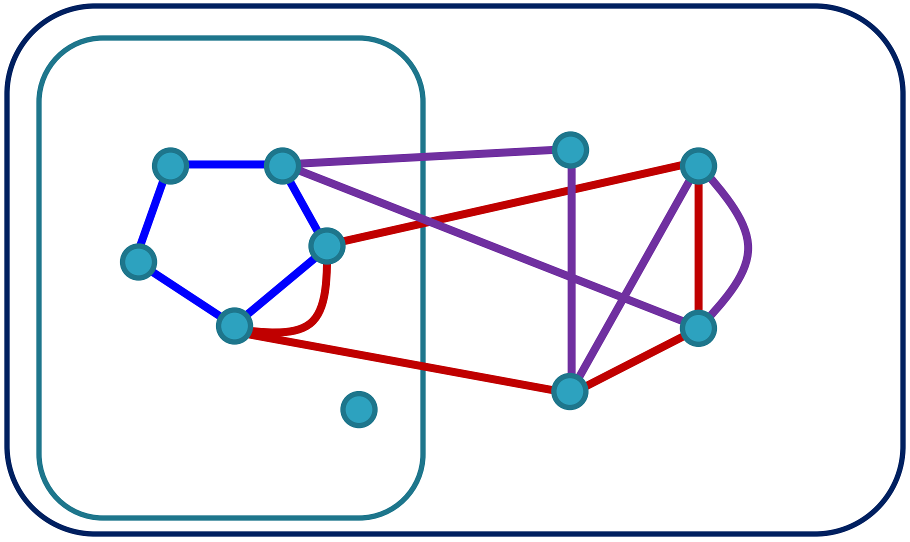
- The Lindner problem of embedding a partial 3-cycle system of order n into an STS(v) has been solved! (Bryant and Horsley)
  - A necessary condition requires that $v \geq 2n+1$
- For 4-cycles the situation is messier, but recent progress has been dramatic:
  - Necessarily $v \geq n+n^{1/2}-1$
  - Lindner had the best result of 2n+15 until recently:
    - $n + 12^{1/2}n^{3/4} + o(n^{3/4})$ (Lindner and Hilton)
    - $n + n^{1/2} + o(n^{1/2})$ (Füredi and Lehel)

Embeddings - History

- Partial 3-cycle system of order n into an STS(v)
  - $v \geq 2n+1$ (Bryant, Horsley)
- Partial 4-cycles systems:
  - $n + n^{1/2} + o(n^{1/2})$ (Füredi, Lehel)
- Partial 5-cycle systems:
  - $(9n + 146)/4$ (Martin, McCourt)
- Partial 2k-cycle systems
  - Around kn (Hoffman, Lindner ,Rodger)
- Partial 2k+1-cycle systems
  - Around (4k+2)n (Lindner, Rodger, Stinson)

Enclosings

- A k-cycle system P of λK_v is said to be enclosed in a k-cycle system Q of $(\lambda+\mu)K_{v+u}$ if P is a sub-multiset of Q.



$k = 5$

P with $\lambda = 1$ Q with $\lambda+\mu = 2$

Conjecture

A 5-cycle system of λK_v can be enclosed in a 5-cycle system of $(\lambda+\mu)K_{v+u}$ if and only if

1. $(\lambda+\mu)(v+u-1)$ is even,
2. The number of new edges is divisible by 5,
3. If $u = 1$ then $\mu(v-1) \geq 3(\lambda + \mu)$,
4. If $u = 2$ then

   $\mu v(v-1)/2 - 2(\lambda + \mu) - (v-1)(\lambda + \mu)/2 \geq 0$, and
5. If $u \geq 3$ then

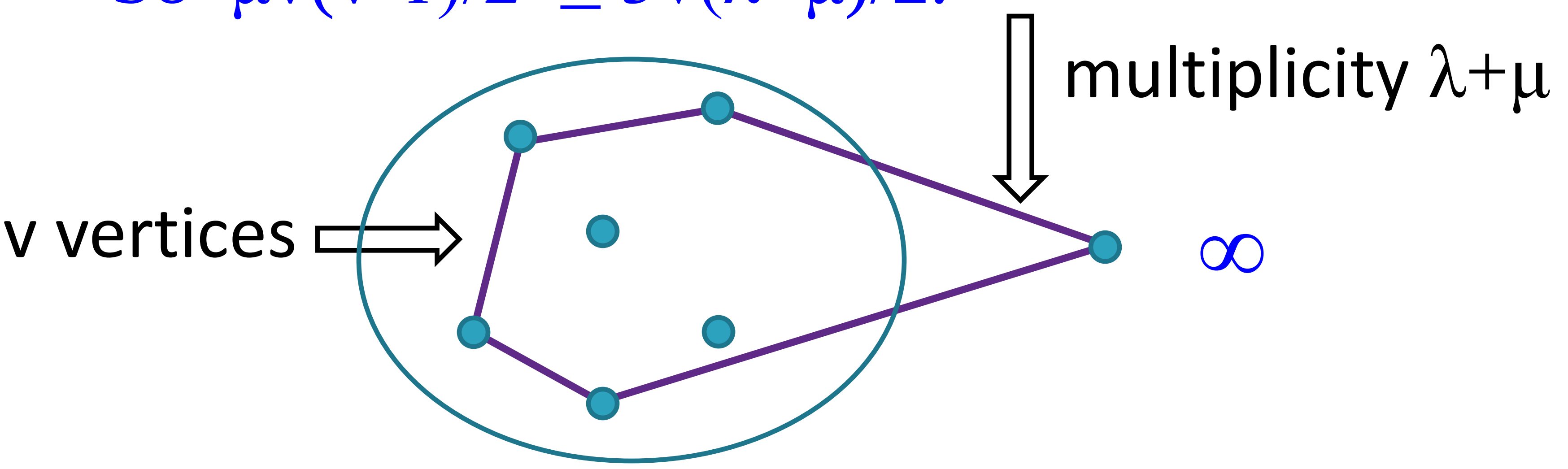
   $\mu v(v-1)/2 + (\lambda + \mu)u(u-1)/2 \geq vu(\lambda + \mu)/4 + 2\varepsilon$

where $\varepsilon = 0$ or 1 if $vu(\lambda + \mu)$ is 0 or $2 \pmod 4$ resp.

(Asplund, Keranen and Rodger)

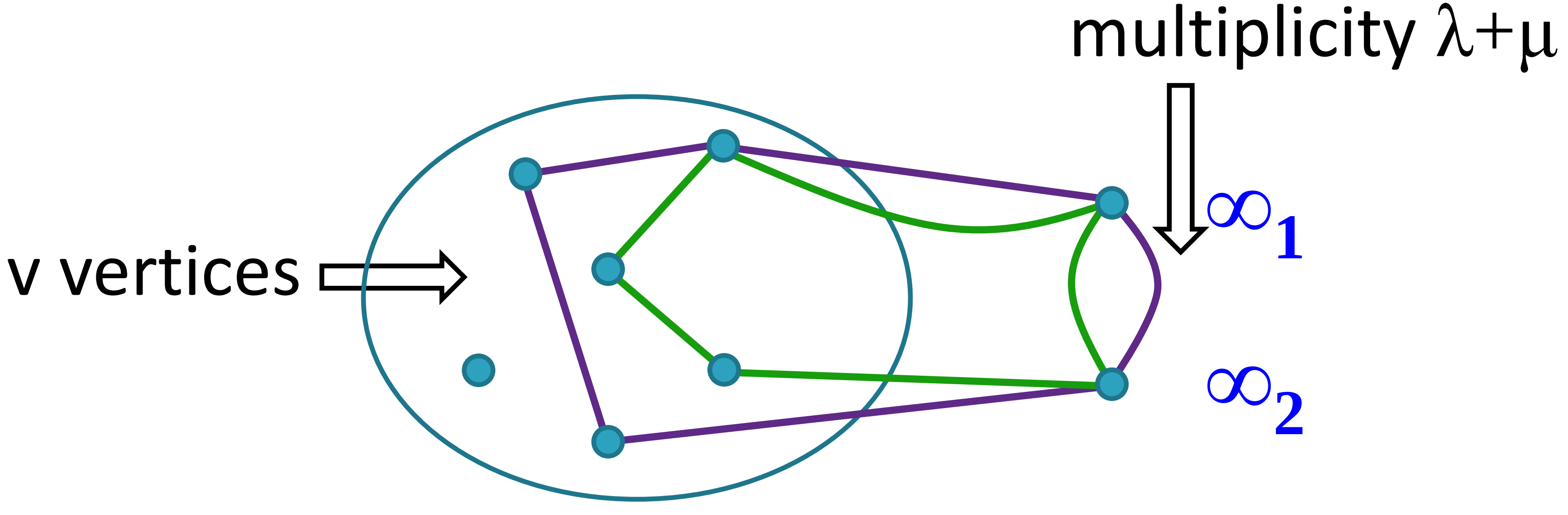
These Conditions are Necessary

- Suppose u = 1.
- The number of 5-cycles including the added vertex, ∞, must be $v(\lambda+\mu)/2$.
- Each of these uses 3 edges in K_v.
- So $\mu v(v-1)/2 \geq 3v(\lambda+\mu)/2$.



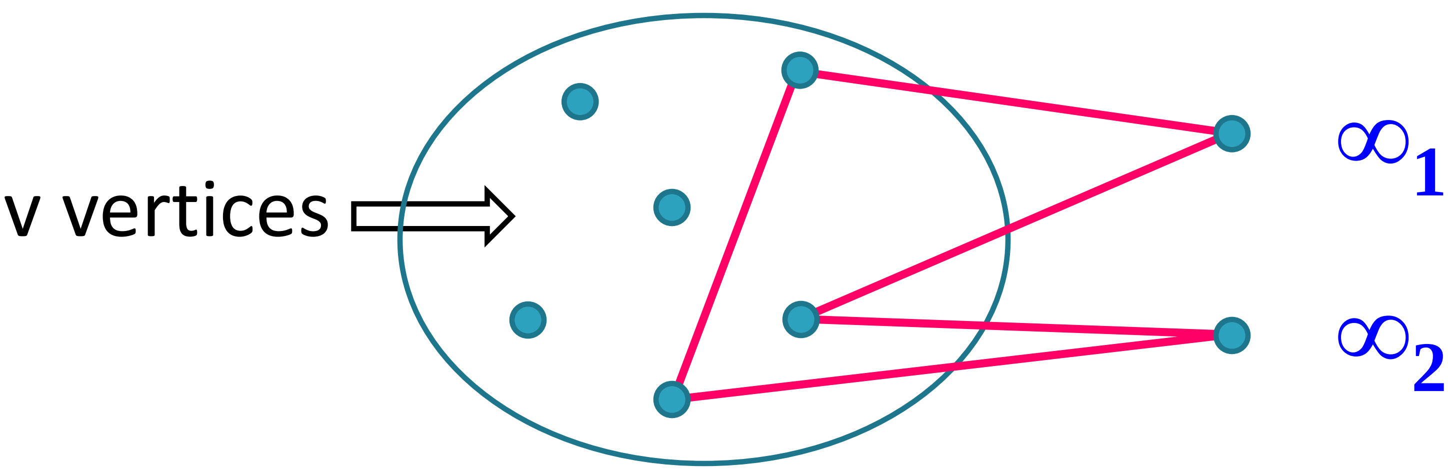
These Conditions are Necessary

- Suppose u = 2.
- The number of 5-cycles joining the two added vertices must be $(\lambda+\mu)$.
- Each of these uses exactly 2 edges in K_v.



These Conditions are Necessary

- Suppose u = 2.
- The number of remaining edges joining the 2 new vertices to K_v is $2v(\lambda+\mu) - 2(\lambda+\mu)$
- Each of the 5-cycles using these $2(v-1)(\lambda+\mu)$ edges uses at least 1 edge in K_v.

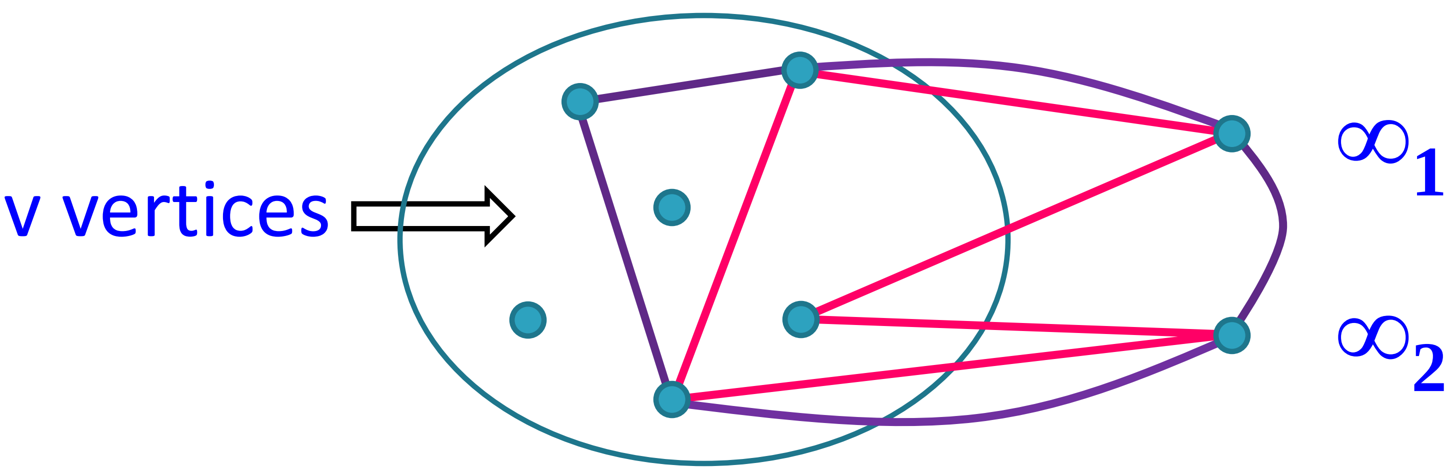


These Conditions are Necessary

- Suppose u = 2.
- So

$$2(\lambda+\mu) + 2(v-1)(\lambda+\mu)/4 \leq \mu v(v-1)/2$$

- $(v-1)(\lambda+\mu)$ is even



Theorem

A 5-cycle system of λK_v can be enclosed in a 5-cycle system of $(\lambda+\mu)K_{v+1}$ if and only if

1. $(\lambda+\mu)(v+u-1)$ is even,
2. The number of new edges is divisible by 5, and
3. $\mu(v-1) \geq 3(\lambda + \mu)$.

(Asplund, Keranen and Rodger)

An idea of the proof

Good news!

The number of edges that occur in 5-cycles completely contained in the μK_v is exactly

$$\mu v(v-1)/2 \ - 3v(\lambda + \mu)/2 = \alpha v$$

so is a multiple of v. (α is always an integer)

$(3(\lambda + \mu)/2)v$ edges occur in 3-paths.

We are in with a chance of using difference methods!

An example will suffice!
$v = 50, \mu = 1$

So the necessary condition

$$\mu(v-1) \geq 3(\lambda + \mu)$$

means that

$$\lambda \leq \mu(v-4)/3 = 15.3$$

so

$$\lambda \leq 14.$$

We start with the small values and work our way up.

Skolem Sequences

The two integers, *k*, appear *k* apart in:

<u>1</u> <u>1</u> <u>3</u> <u>4</u> <u>2</u> <u>3</u> <u>2</u> <u>4</u>

1 2 3 4 5 6 7 8

These can be represented by pairs:

{1,2} {3,6} {4,8} {5,7}

Or we can add a constant to each number in each pair:

{4,5} {6,9} {7,11} {8,10}

v = 50=10k, μ = 1, λ = 2: so α = 20

{4,5} {8,10} {6,9} {7,11}

Differences used:

4,5,8,**10**,6,9,7,11=2k+1

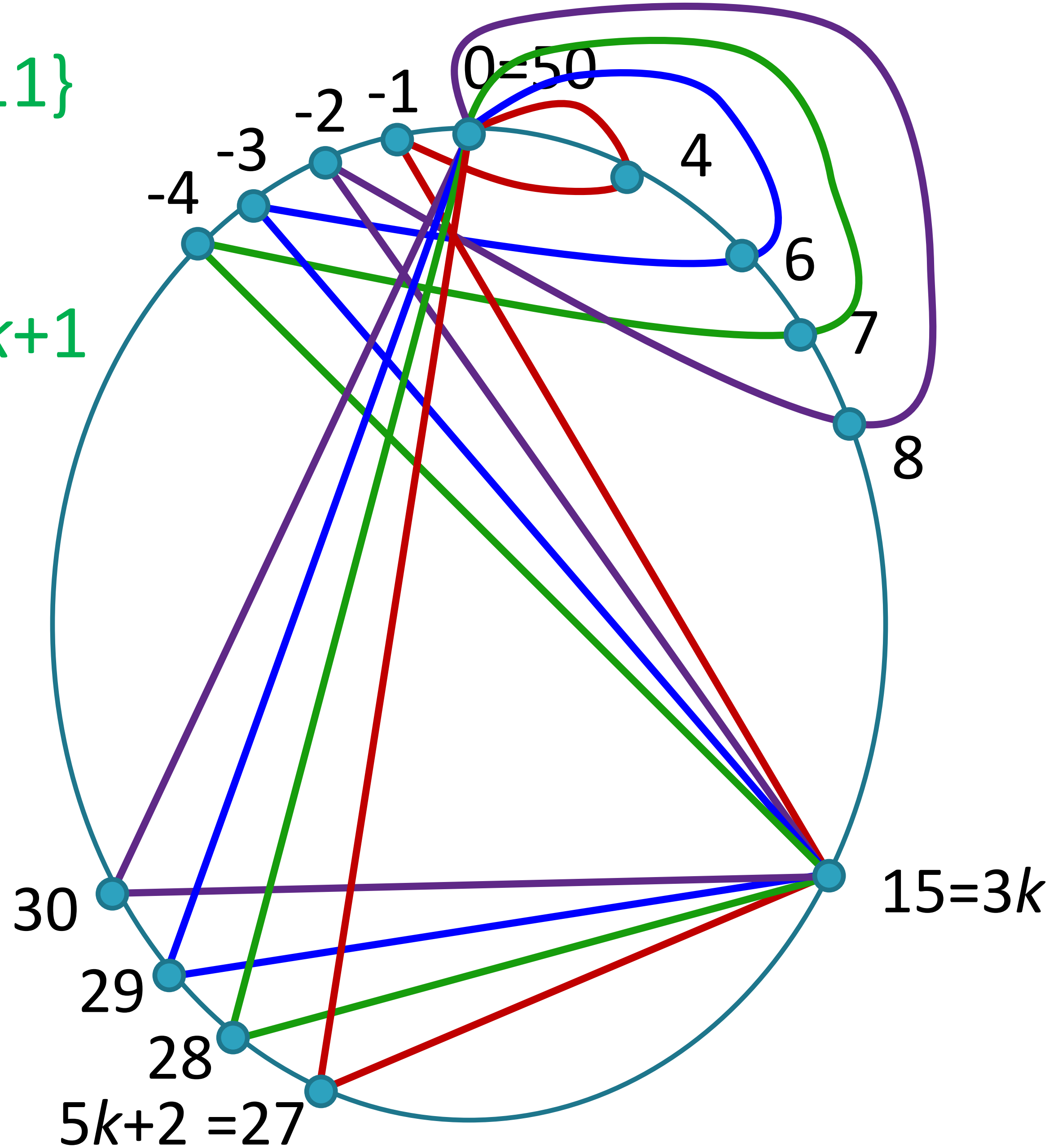
16, 17, 18, 19

12, 13, 14, 15

23, 22, 21, **20**=4k

What is left??

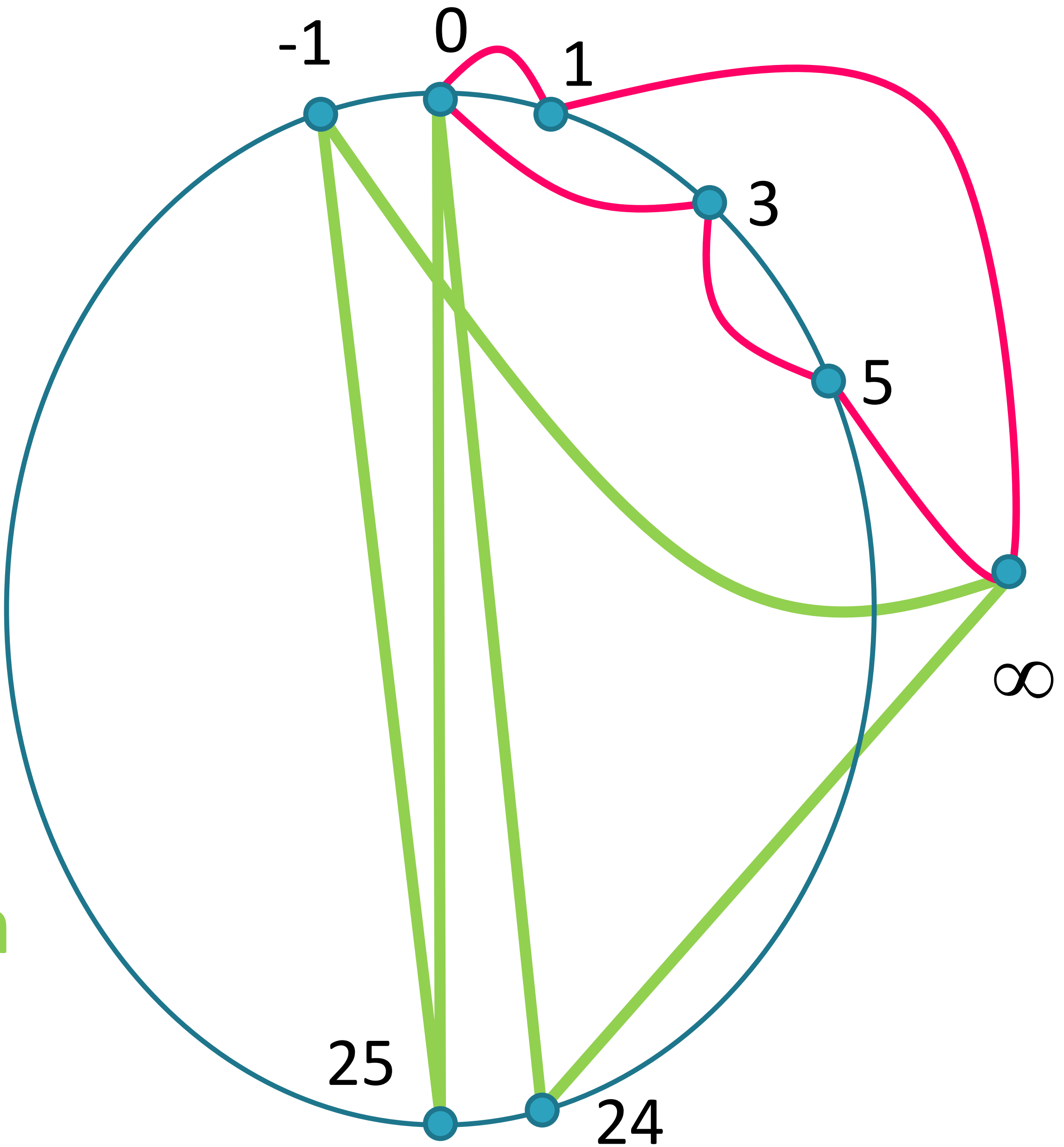
1, 2, 3, 24, 25



What do we do with differences 1,2,3,24,25?

These edges occur
In 5-cycles with ∞

Rotate this through 50 positions: $\lambda = 2$

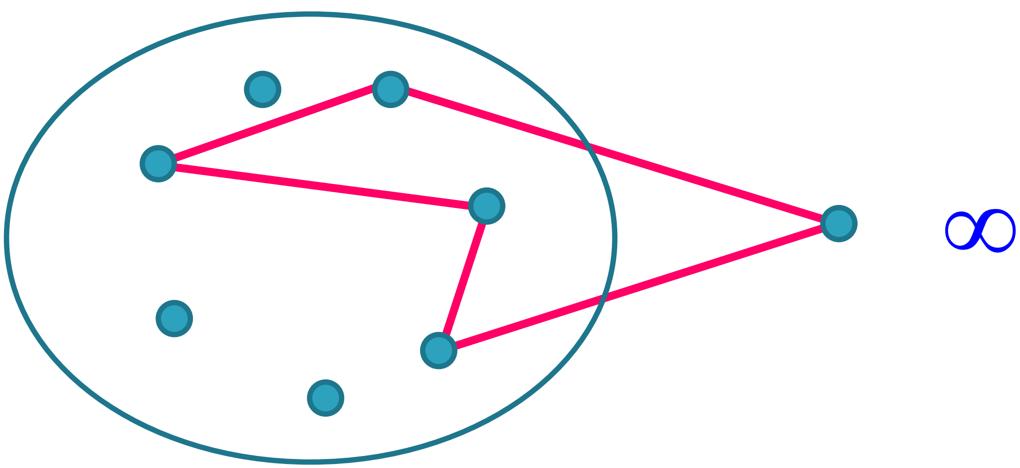


Rotate this through 25 positions: $\mu = 1$

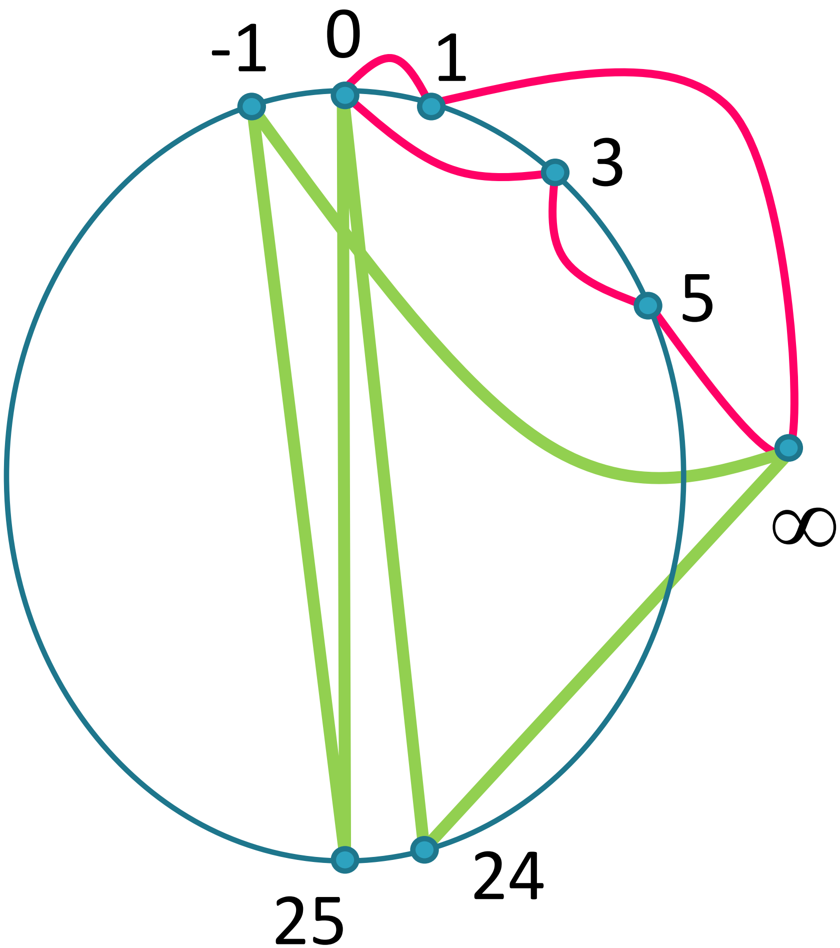
$v = 50$, $\mu = 1$,

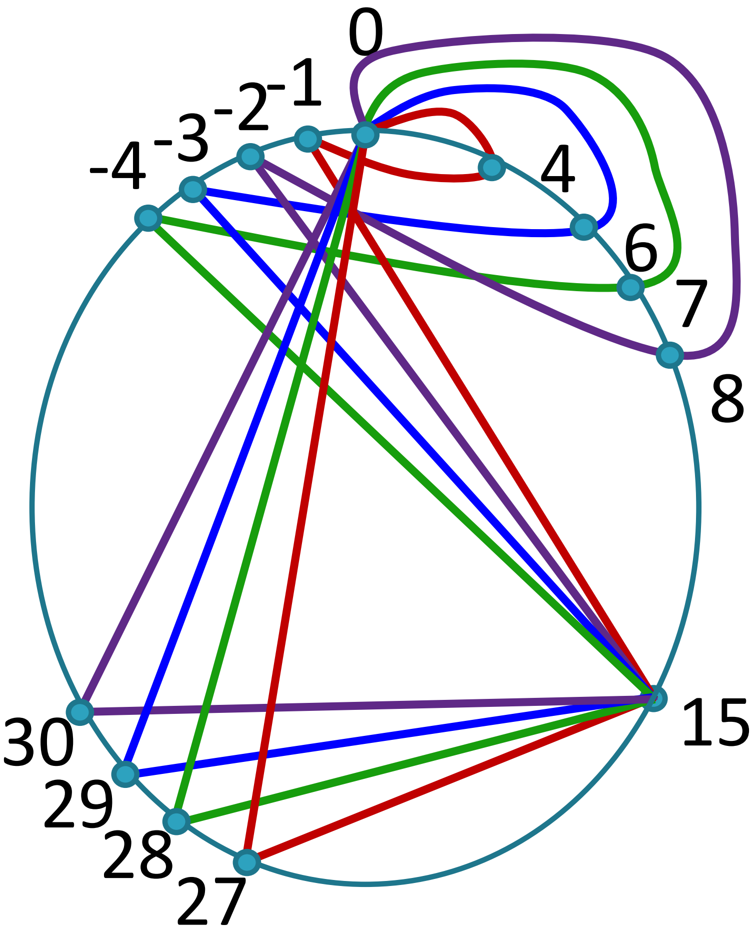
What happens if λ is bigger?

- λ must be even
- $\alpha = \mu(v-1)/2 \; - 3(\lambda + \mu)/2$
- So increasing λ by 2 means that α is decreased by 3



v = 50, μ = 1, λ = 4: so α = 17

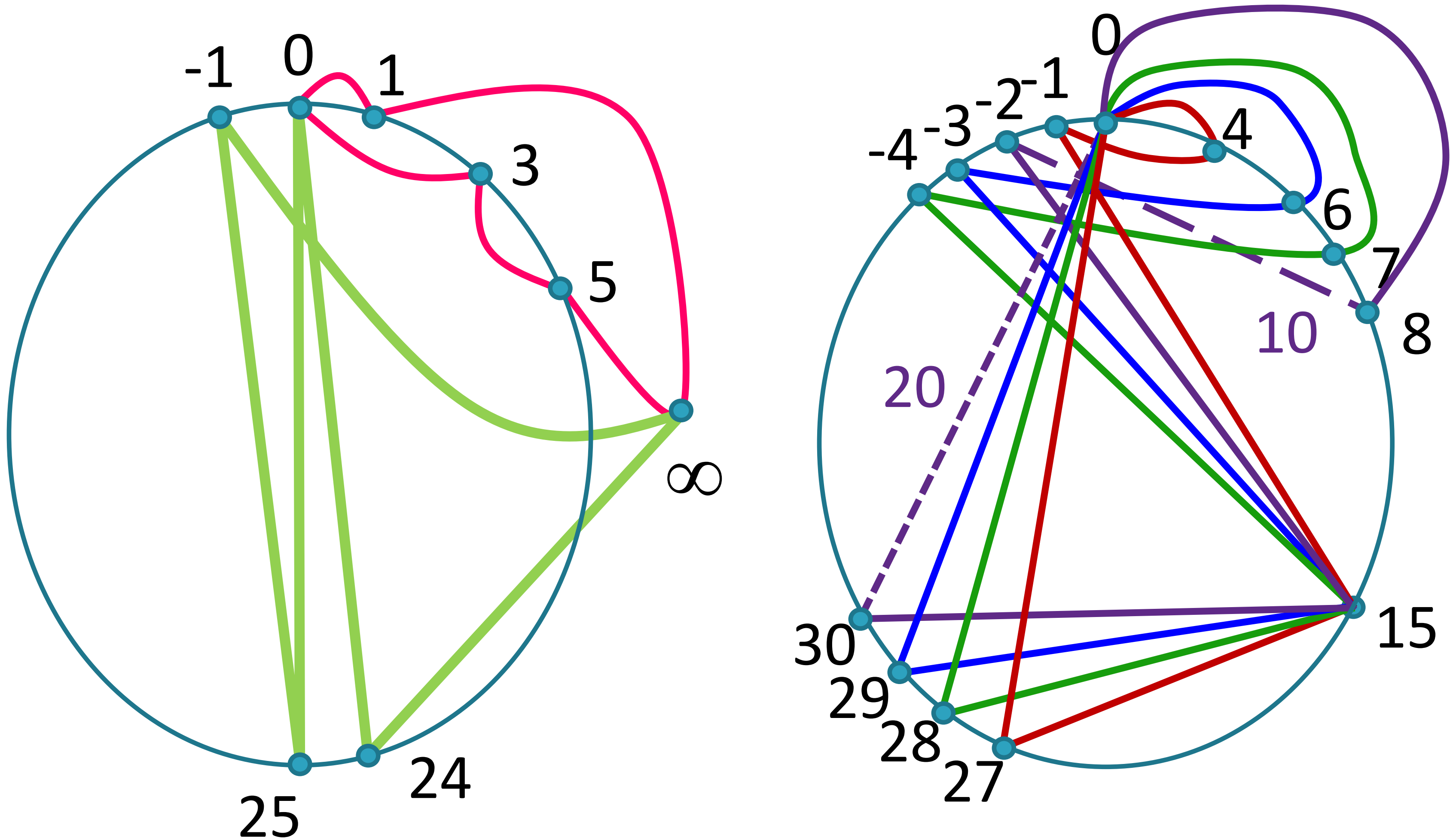




How do we drop to 17 differences on the right??

Look at the purple 5-cycle: 8, **10**, 13, 17, **20**

$v = 50, \mu = 1, \lambda = 4$: so $\alpha = 17$



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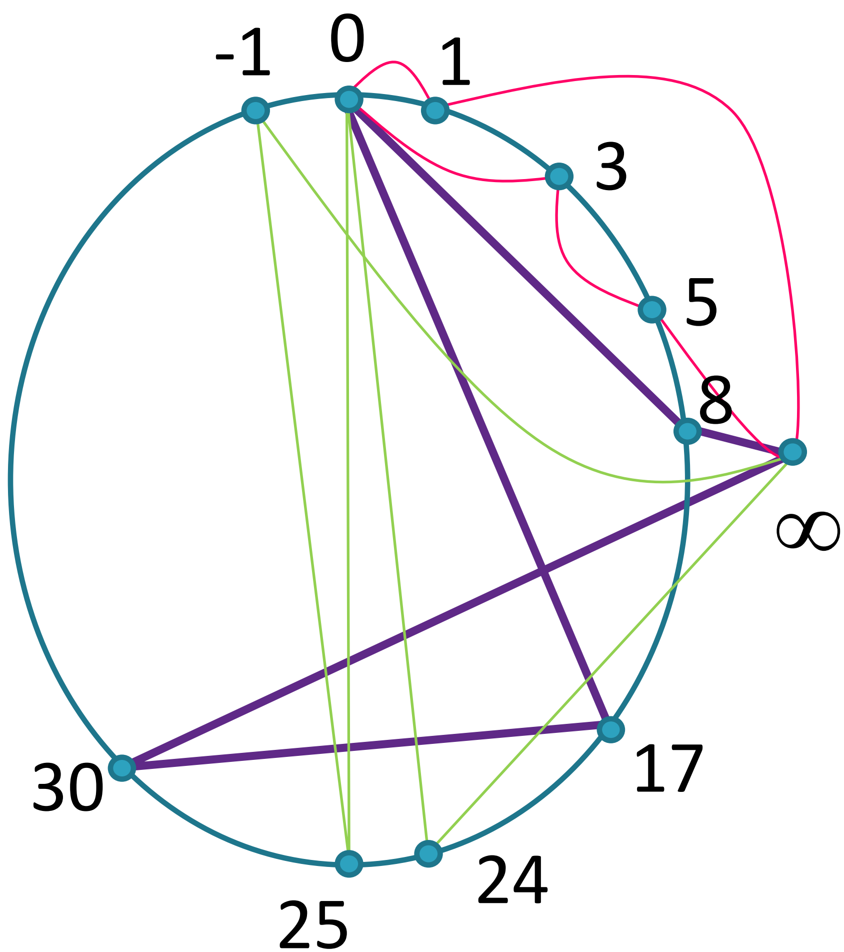
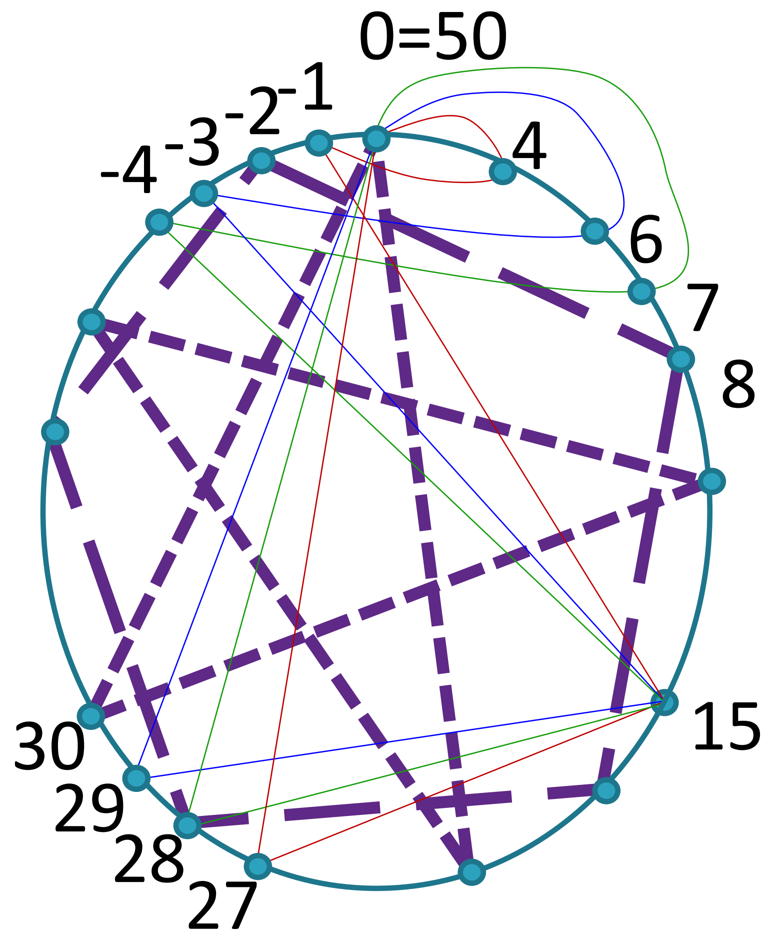




The purple 5-cycle pieces:

8, 13, 17

Two more purple 5-cycle pieces:

10, **20**

What happens when $\lambda = 6$; so $\alpha = 14$??

- When $\alpha = 14$, somehow we need to use edges of 4 differences and partition them into 5-cycles!
- We can use edges of difference 10 and 20, like before, but we don't have 2 more options.
- The edges of differences 1,2, and 3 can be partitioned into sets that induce 5-cycles!

Remember the Skolem Sequences??

The two integers *k* appear *k* apart in:

<u>1</u> <u>1</u> <u>3</u> <u>4</u> <u>2</u> <u>3</u> <u>2</u> <u>4</u>

1 2 3 4 5 6 7 8

These can be represented by pairs:

{1,2} {3,6} {4,8} {5,7}

Or we can add a constant to each number in each pair: Why add 3??

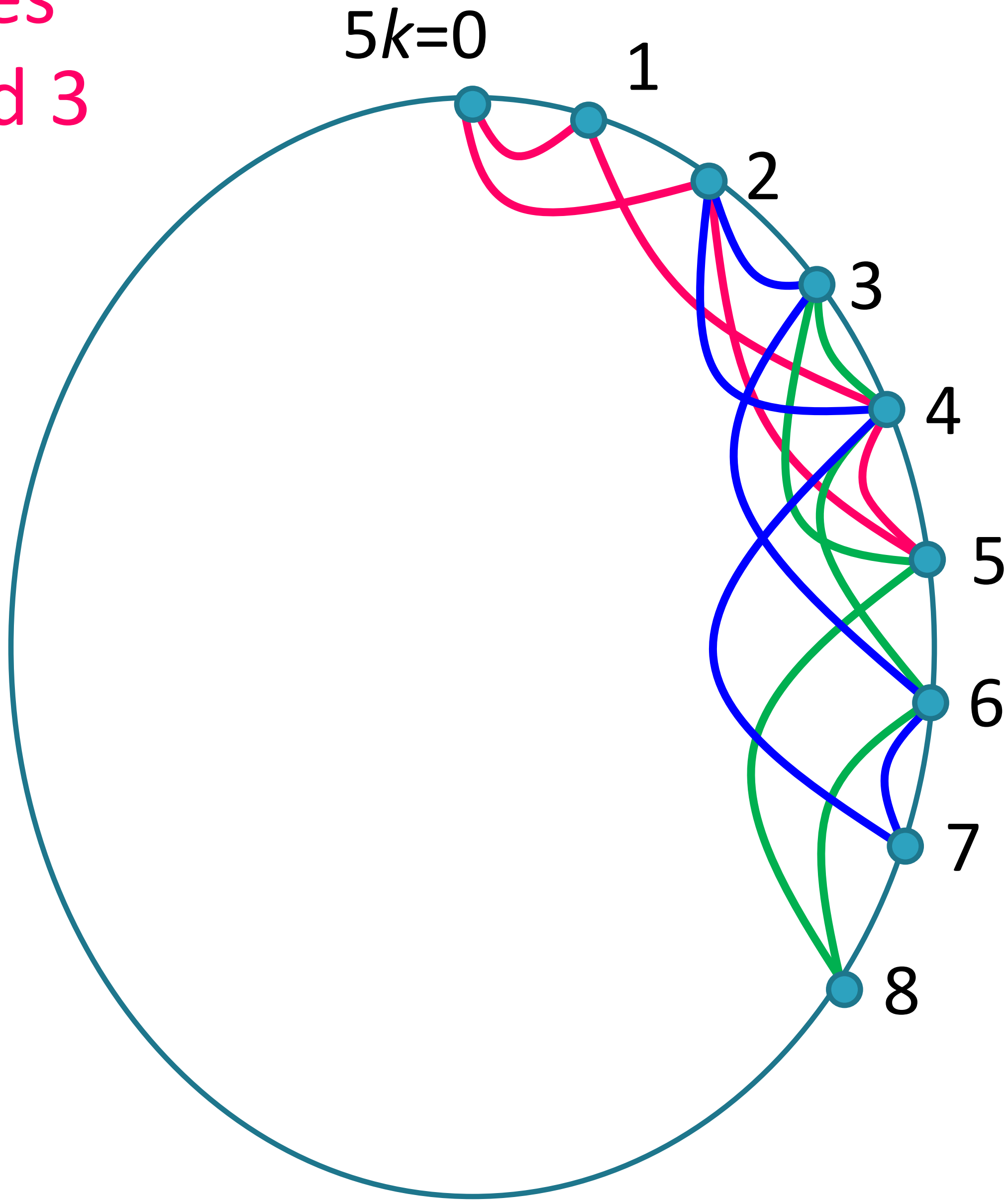
{4,5} {6,9} {7,11} {8,10}

So we avoid using differences 1,2 and 3 in the 5-cycles in K_{50}!

Notice that only edges of differences 1,2 and 3 are used.

Here is the second of three base cycles.

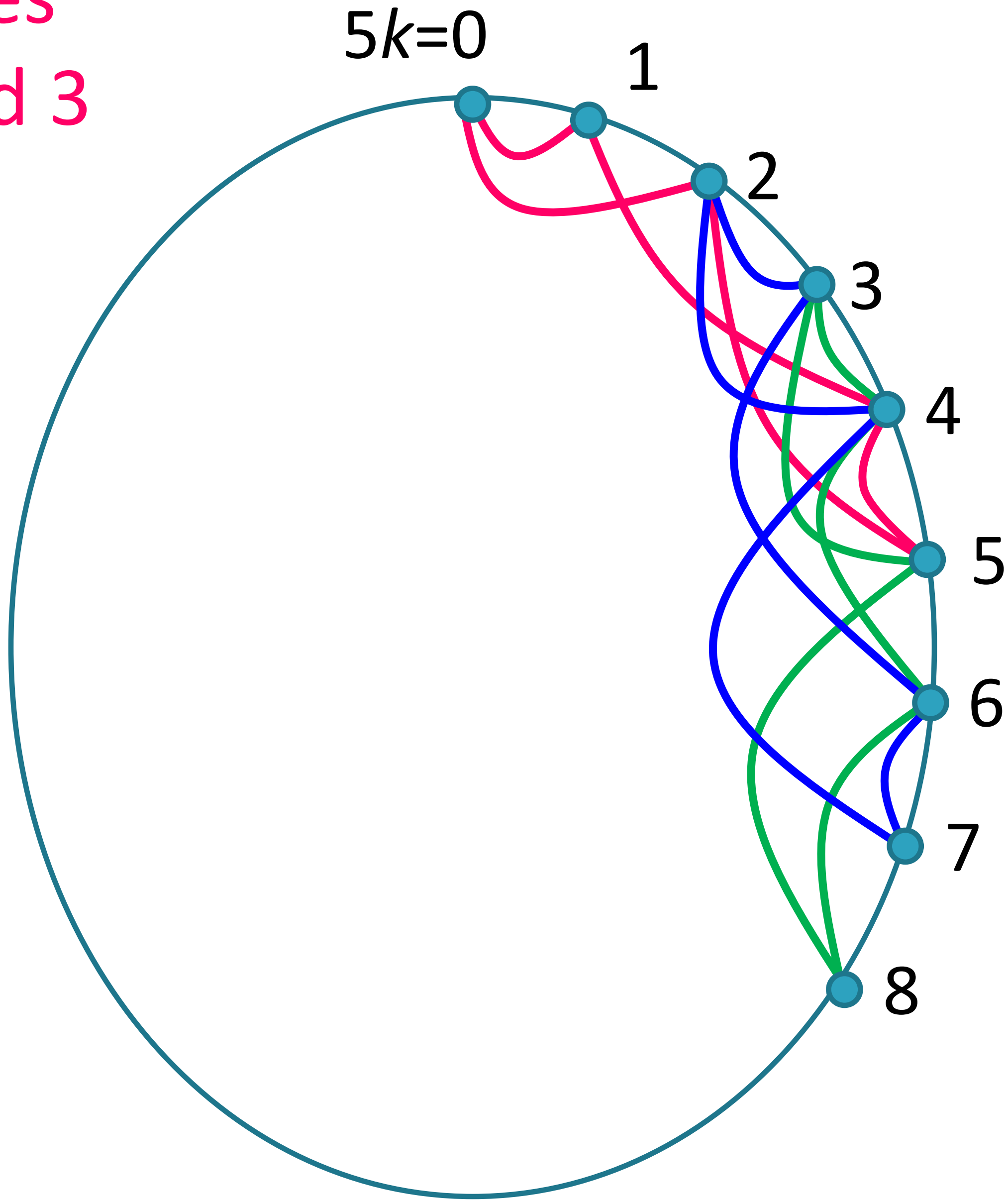
Each of these will have multiples of 5 added to them.



Notice that only edges of differences 1,2 and 3 are used.

Look at the edges of difference 1 in these cycles.

Each of these will have multiples of 5 added to them.



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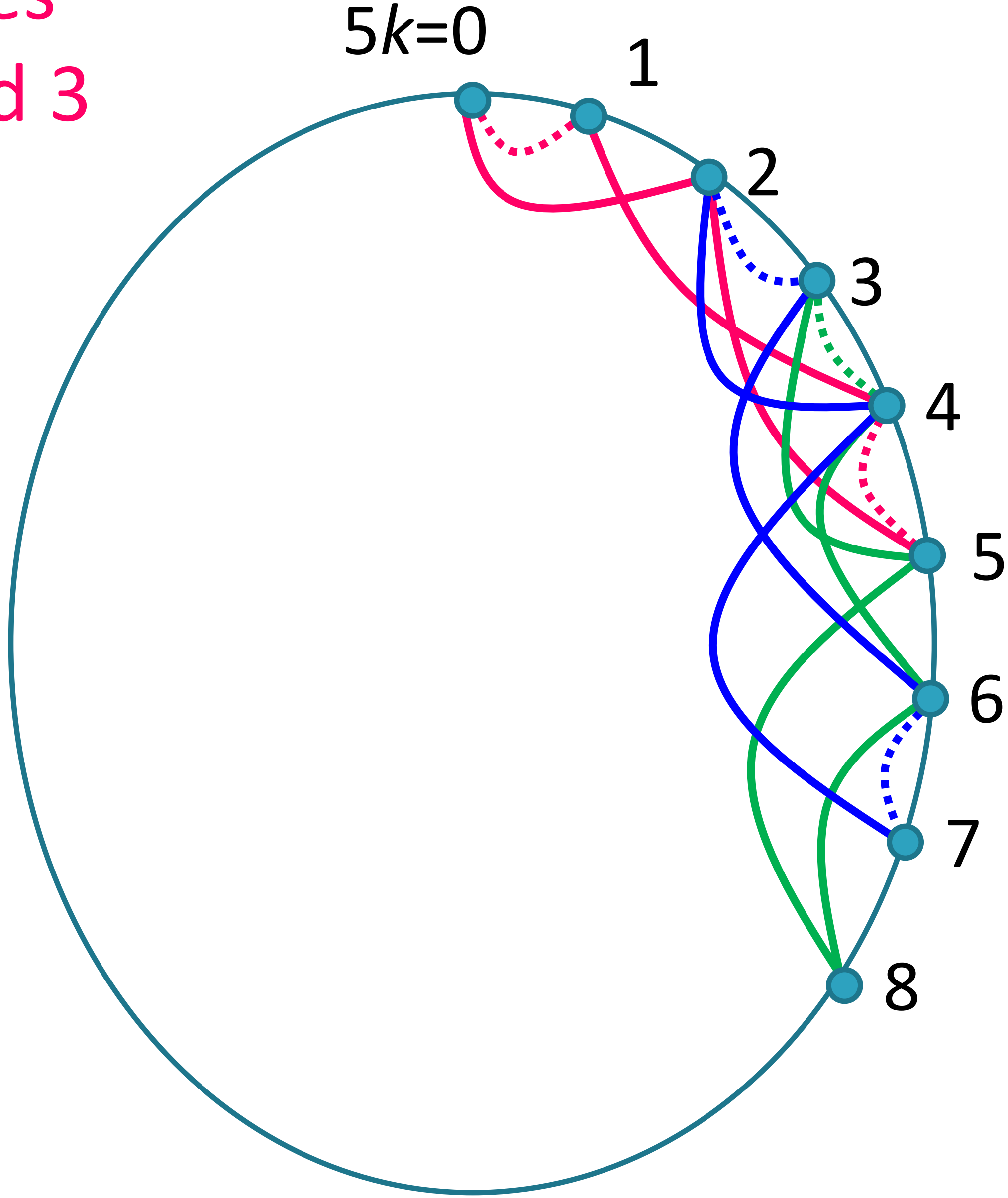
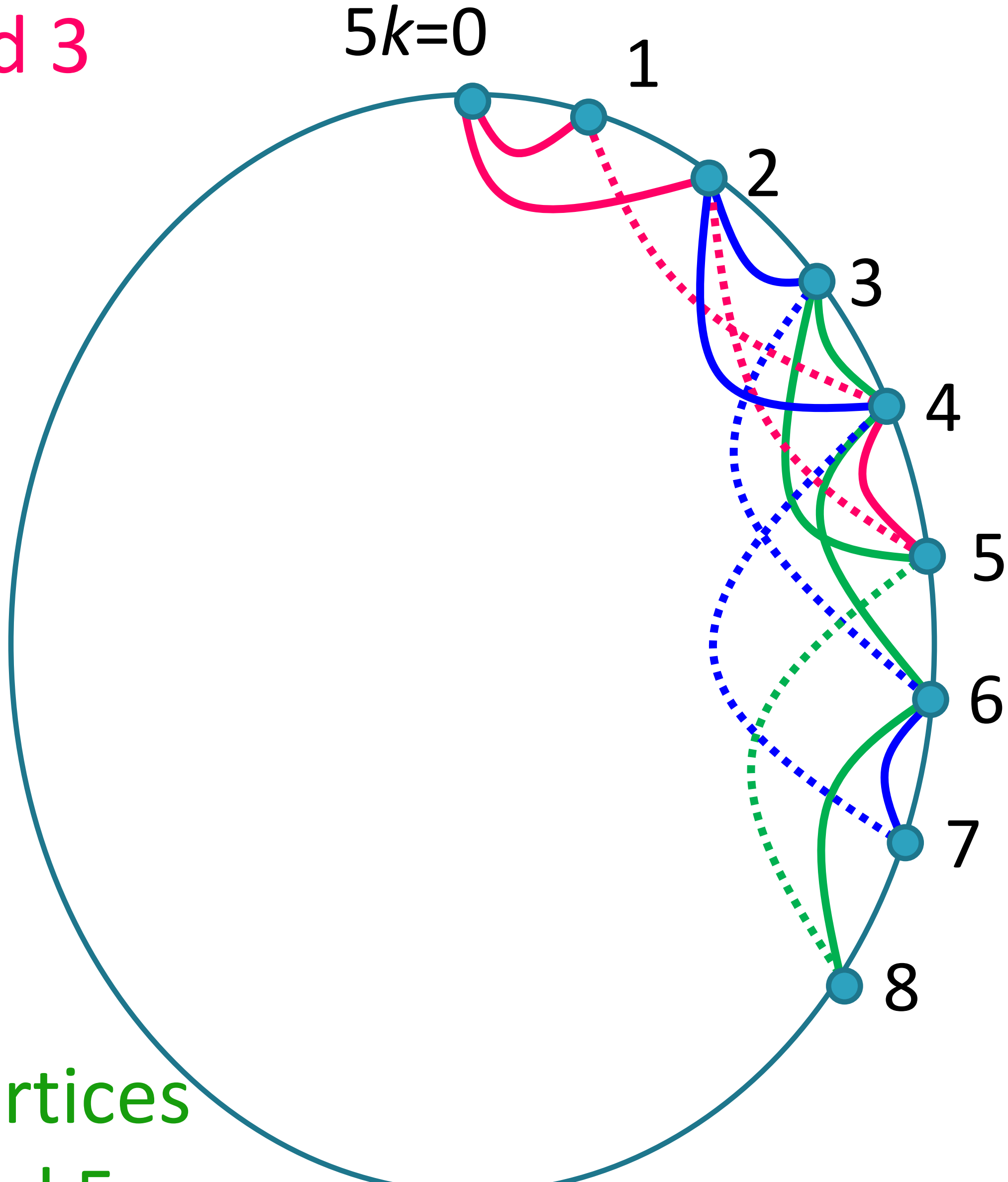


Notice that only edges of differences 1,2 and 3 are used.

Look at the edges of difference 1 in these cycles.

Each of these will have multiples of 5 added to them.

The 5 edges of each difference start at vertices that are different mod 5.



v = 50, $\mu = 1$, $\lambda = 6$: so $\alpha = 14$

{4,5} {8,10} {6,9} {7,11}

Differences used:

4,5,12,16,23

6,9,14,18,21

10,

1,2,3

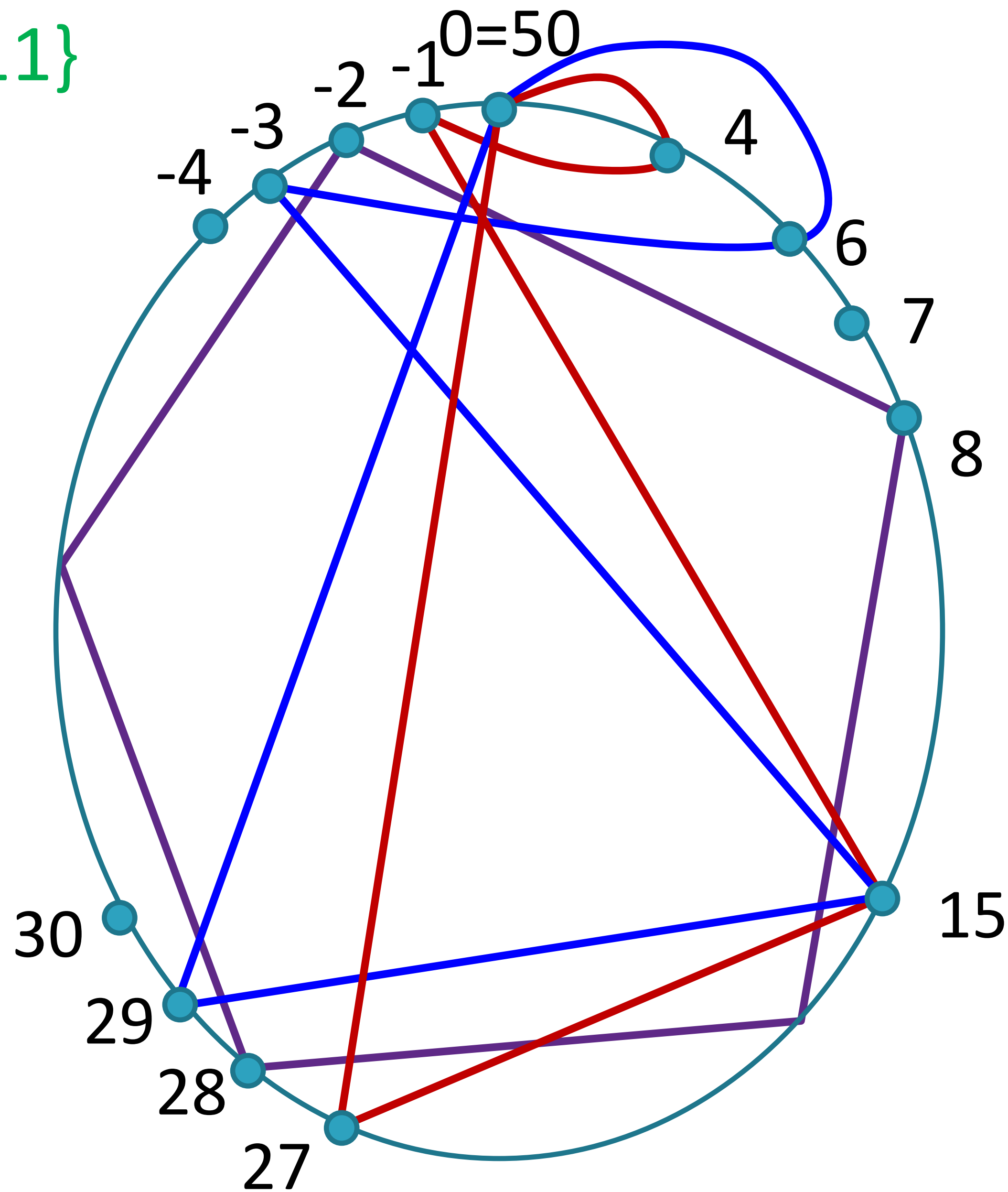
What is left??

Group these in 3's:

8,15,17,20

7,11,13,19,22

24,25 gives $\mu = 1$



What happens when μ increases?

When $v = 50$ and $\mu = 1$, $\lambda \leq 14$.

When $\mu = 2$, the necessary condition

$$\mu(v-1) \geq 3(\lambda + \mu)$$

means that

$$\lambda \leq \mu(v-4)/3 = 30.6$$

so

$$\lambda \leq 30.$$

So every case can be handled using the $\mu = 1$ result

EXCEPT WHEN $\lambda = 30$!!

Is this v=50 case typical?

- For the most part, yes.
- The smallest value of λ is also a problem:

μ: v (mod 10)	1	2	3	4	5	6	7	8	9	10
1	9	8	7	6	5	4	3	2	1	0
6	4	8	2	6	0	4	8	2	6	0

Theorem (in all likelihood!)

A 5-cycle system of λK_{v} can be enclosed in a 5-cycle system of $(\lambda+\mu)K_{v+2}$ if and only if

1. $(\lambda+\mu)(v+u-1)$ is even,
2. The number of new edges is divisible by 5, and
3. If $u = 2$ then

$$\mu v(v-1)/2 - 2(\lambda + \mu) - (v-1)(\lambda + \mu)/2 \geq 0$$

(Asplund, Keranen and Rodger)

Conjecture

A 5-cycle system of λK_v can be enclosed in a 5-cycle system of $(\lambda+\mu)K_{v+u}$ with $u \geq 3$ if and only if

1. $(\lambda+\mu)(v+u-1)$ is even,
2. The number of new edges is divisible by 5, and
3. $\mu v(v-1)/2 + (\lambda + \mu)u(u-1)/2 \geq vu(\lambda + \mu)/4 + 2\varepsilon$

where $\varepsilon = 0$ or 1 if $vu(\lambda + \mu)$ is 0 or 2 (mod 4) respectively.

(Asplund, Keranen and Rodger)

There is a gap!

$$\mu v(v-1)/2 + (\lambda + \mu)u(u-1)/2 \geq vu(\lambda + \mu)/4 + 2\varepsilon \qquad *$$

is quadratic in v.

So as v increases with the other 3 parameters held constant, enclosings may become impossible for some interval, then become possible again.

For example when $\mu = 1$, $\lambda = 34$ and $u = 7$, * requires:

$$v \leq 10 \text{ or } v \geq 120$$

2009

2010

2011

2012

Finally – he's in a photo!

Ian Roberts!

Theorem

A 4-cycle system of λK_v can be enclosed in a 4-cycle system of $(\lambda+\mu)K_{v+u}$ if and only if

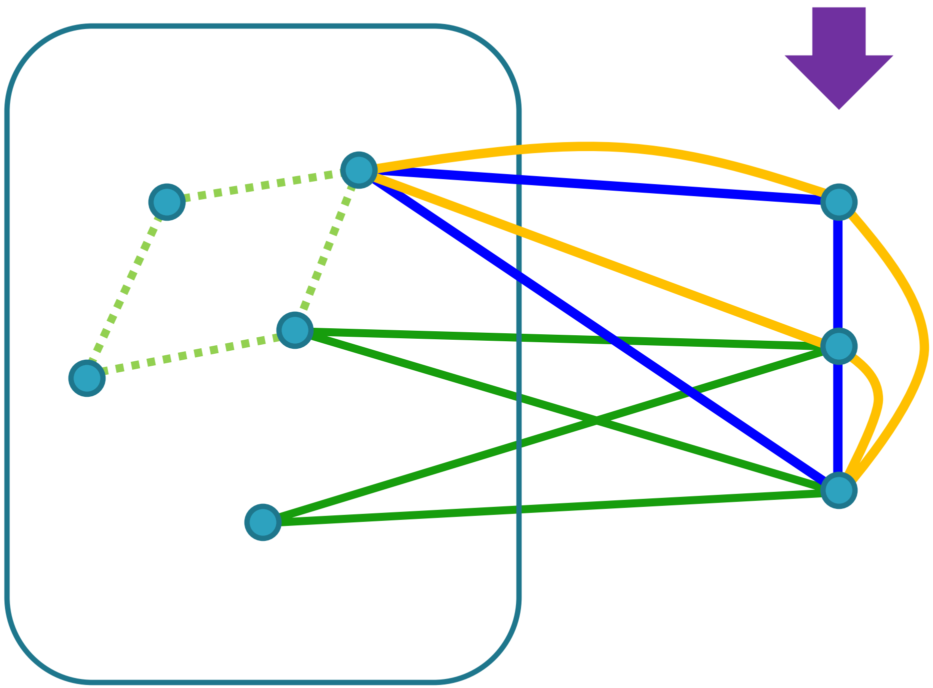
1. $(v+u-1)(\lambda+\mu)$ is even,
2. The number of new edges is divisible by 4,
3. If $u = 1$ then $\mu(v-1)/2 \geq \lambda + \mu$, and
4. If $u = 2$ then $\mu v(v-1)/2 \geq \lambda + \mu$.

(Newman and Rodger)

When u $\geq$ 3 it is not hard to settle.

Use existing results on maximum partial 4-cycle systems.

Add at least 3 vertices



It is easy to use the edges that join 2 new vertices!

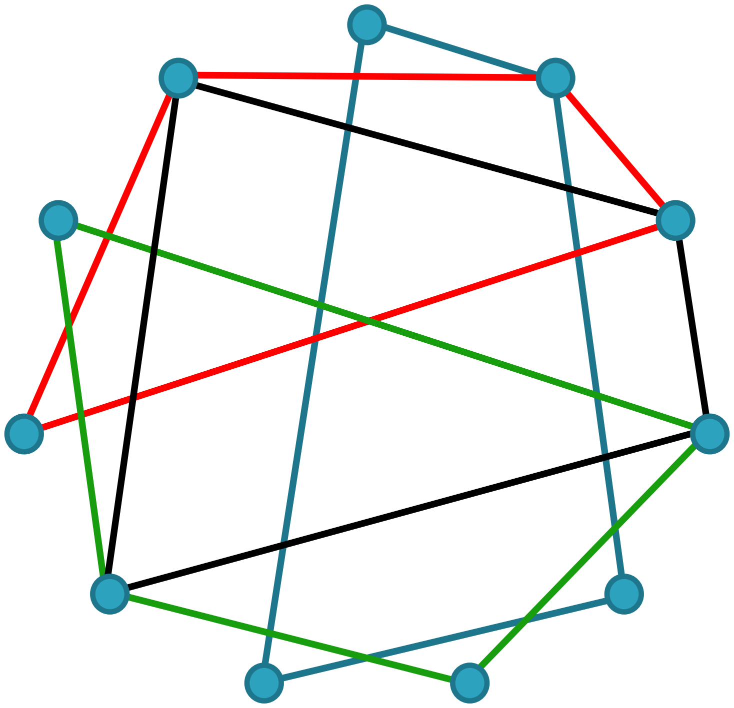
And it's easy to use the other types of edges independently.

Settling u = 2 has a graph theoretic feel

- Solving this case involves:
  - Equitable partial 4-cycle systems,
  - Directed Euler Tours, and
  - Expanding nearly-regular graphs into copies of $K_{2,2}$.

Partial Cycle Systems

- A set of edge disjoint 4-cycles in K_n is said to be a *partial* 4-cycle system of order n.



This is a partial 4-cycle system of order 11 that is EQUITABLE.

Equitable Partial Cycle Systems

- Equitable: for each pair of vertices u and v, the number of cycles containing u differs by at most one from the number of cycles containing v.

These are VERY useful!

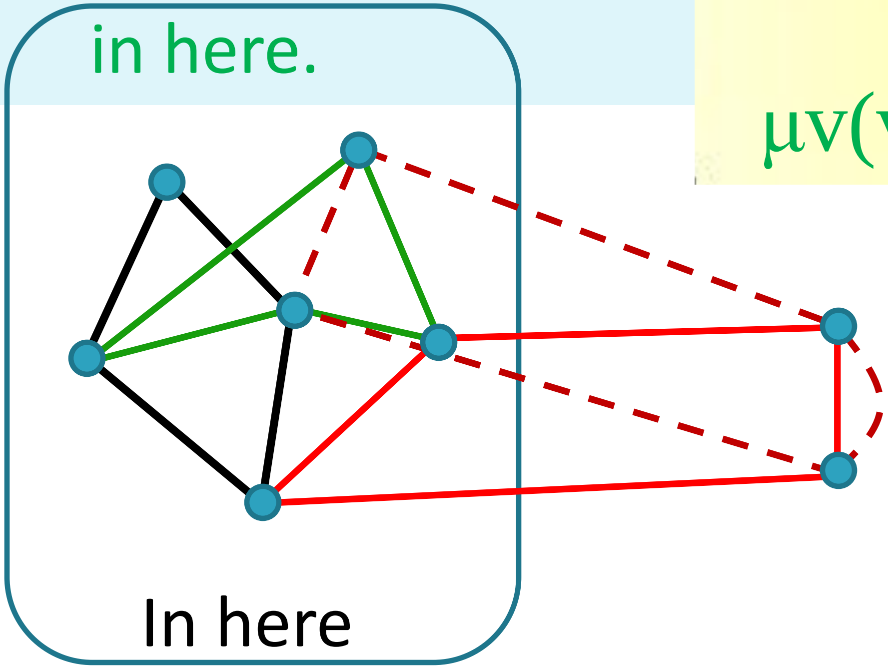
- 3-cycles: Andersen, Hilton and Mendelsohn
- 4-cycles and 5-cycles: Raines and Staniszló
- Any mixture of cycle lengths! Bryant, Horsley and Maenhaut
- When can you partition the edges of K_n into two equitable partial cycle systems of two given lengths (say 3 and 5)?

Recall: If u = 2 then $\mu v(v-1)/2 \geq \lambda + \mu$

Add u = 2 vertices.

Eventually μ more edges are used between each pair of vertices in here.

So
$$\mu v(v-1)/2 \geq (\lambda + \mu)$$

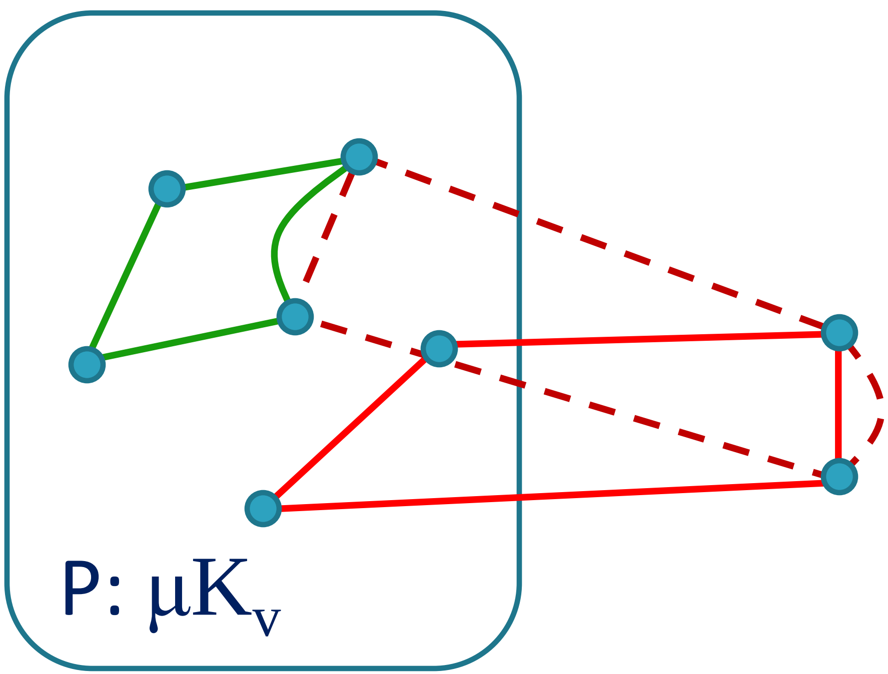


λ edges between each pair of vertices are already used.

There are λ + μ edges joining the 2 added vertices.

Sufficiency with u = 2: $\mu v(v-1)/2 \geq \lambda + \mu$

So exactly $\mu v(v-1)/2 - (\lambda + \mu)$ edges "must" be in 4-cycles joining vertices in P.

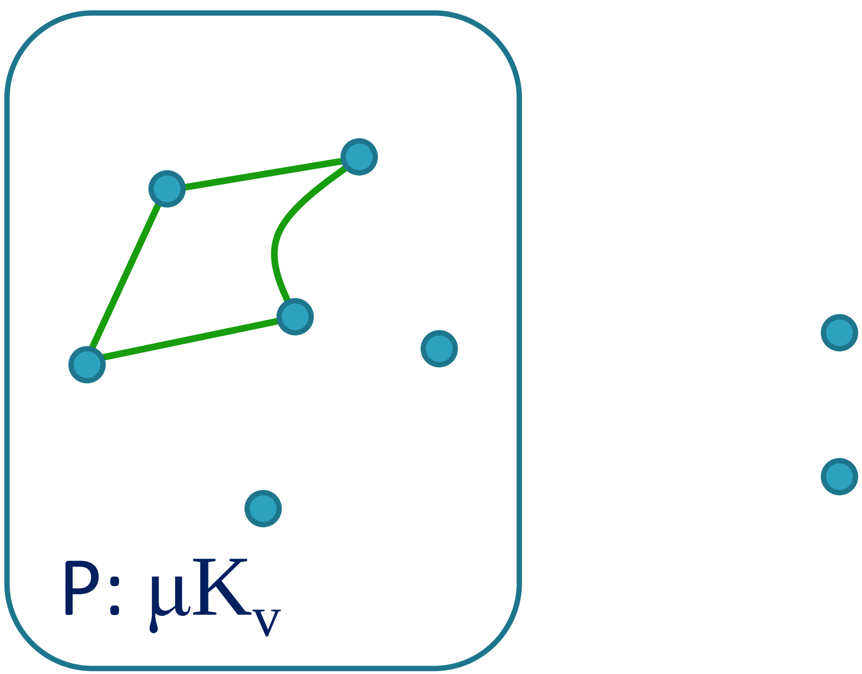


In P, μ edges between each pair of vertices need to be used in 4-cycles.

We just saw: there are $\lambda + \mu$ edges joining the 2 added vertices, each of which must be in 4-cycles like this.

Sufficiency with u = 2: $\mu v(v-1)/2 \geq \lambda + \mu$

So start with an *equitable* partial 4-cycle system C_1 of μK_v with exactly $\mu v(v-1)/2 - (\lambda + \mu)$ edges!



It turns out that this number is divisible by 4.

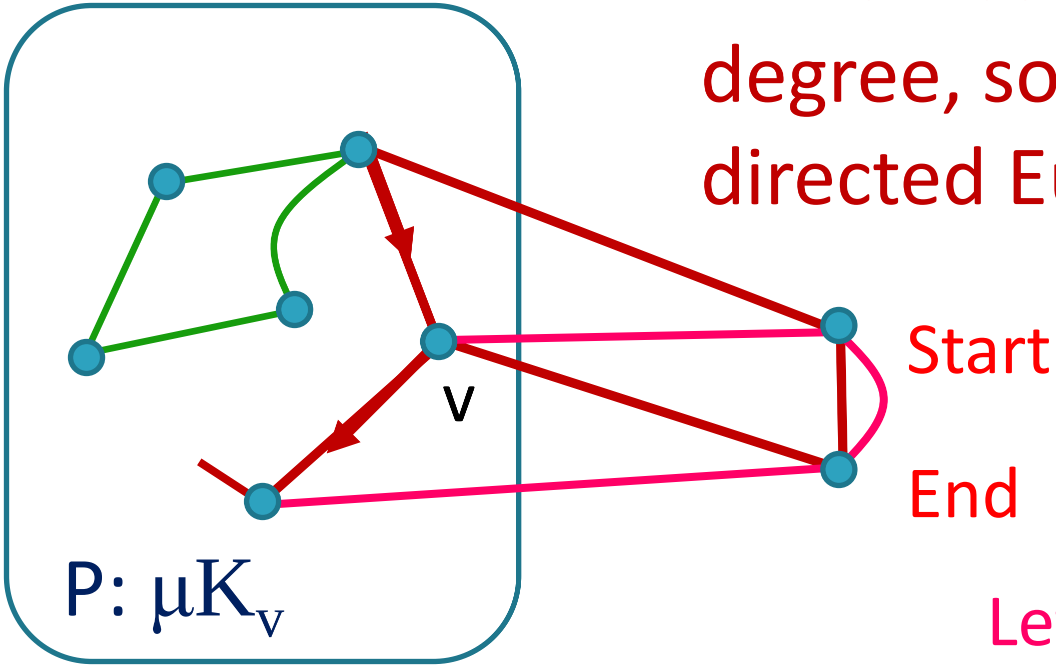
And it is not negative!

Sufficiency with u = 2: $\mu v(v-1)/2 \geq \lambda + \mu$

Now look at the complement in μK_v of C_1.

It has exactly $\lambda + \mu$ edges!

All vertices have even degree, so form a directed Euler tour.



Let C_2 be the set of these 4-cycles.

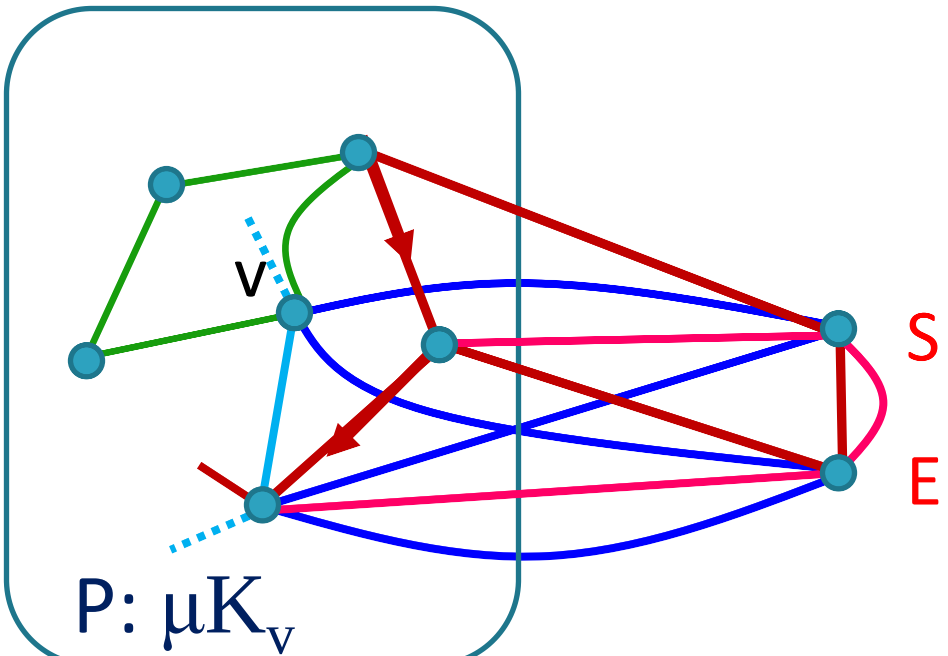
In P, μ edges between each pair of vertices ARE NOW used in 4-cycles.

Sufficiency with u = 2: $\mu v(v-1)/2 \geq \lambda + \mu$

The remaining edges induce a bipartite graph B from P to {S, E}.

Since C_1 is equitable, each vertex v in P has degree 2s or 2s+2 in B (for some s).

Half of the edges incident with v join it to S, half join it to E.



Form a graph on V(P) in which each vertex v has degree $d_B(v)/2$.

For each edge add a 4-cycle.

Other Enclosings

For 3-cycle systems:

- The problem remains open.
- There are earlier results by Colbourn and Hamm, and also with Rosa (South-Eastern Conference in 1985).
- There are several recent results by Hurd, Munson and Sarvate that consider small enclosings.
- One of the necessary conditions is *quadratic*.
- Enclosings do not exist in the interval:

$(v+1)(1-(1-(4mv)/(v-1)^2(\lambda+m)^2)^{1/2}, (v+1)(1+(1-(4mv)/(v-1)^2(\lambda+m)^2)^{1/2}$

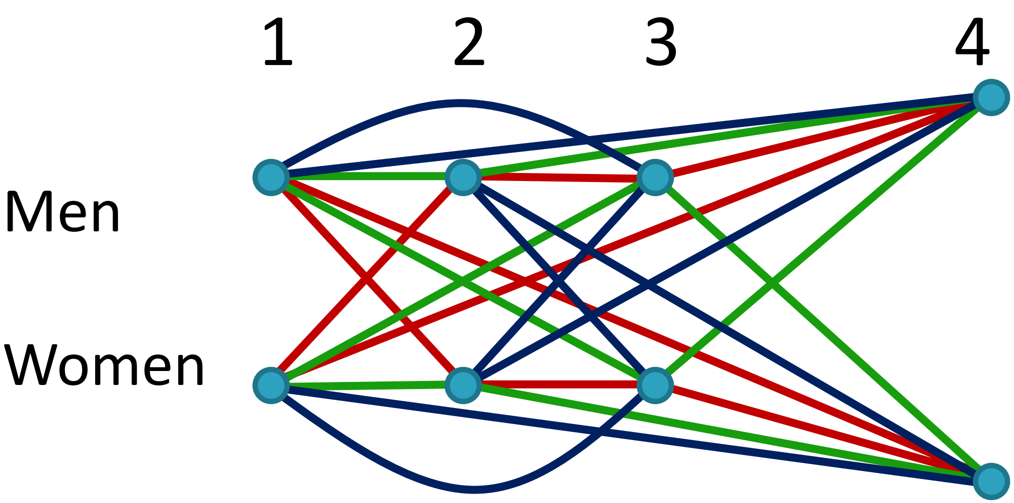
- Recent result approach this gap from both sides (Newman and Rodger

Nothing appears to be known for longer cycles.

Spouse Avoiding Dinners

Try to find a way for 4 couples to sit at 2 tables, each seating 4 people so that each sits next to each other person exactly once.

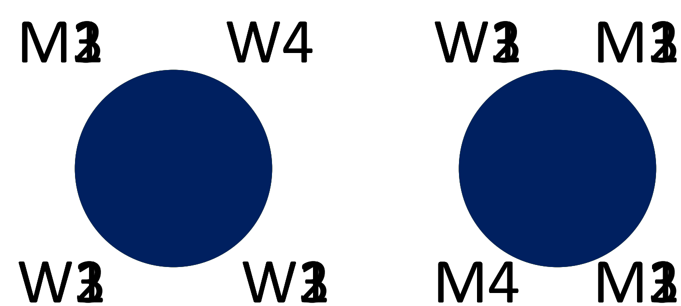
Not the spouses!



Friday

Saturday

Sunday



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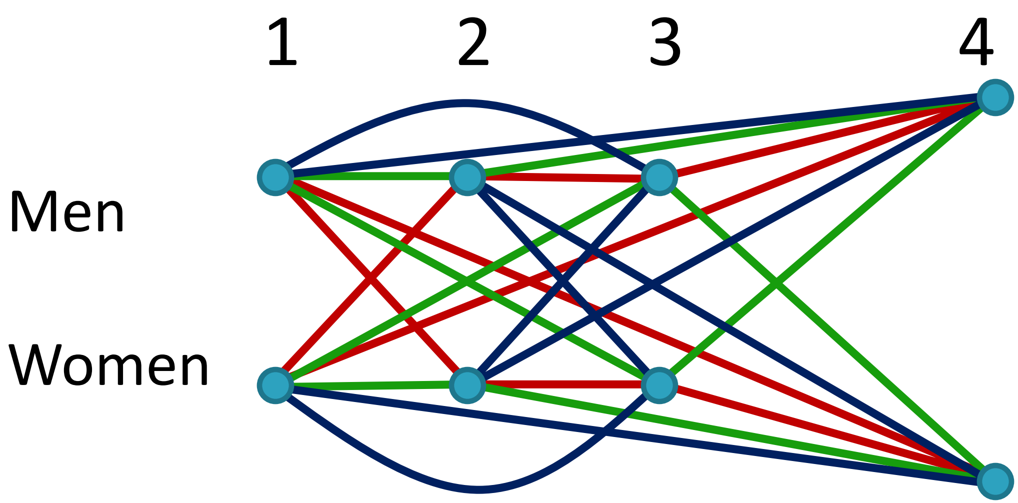


Friday

Saturday

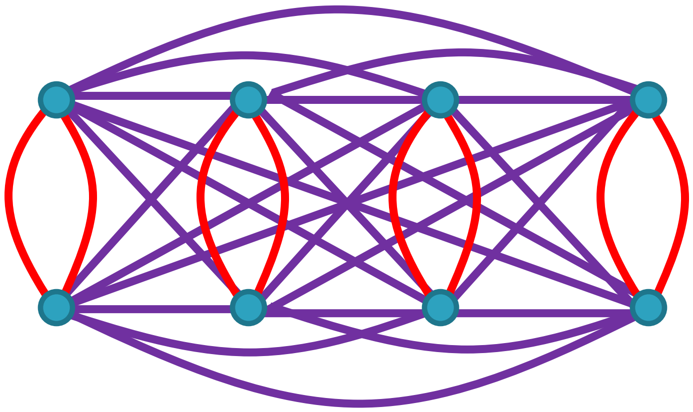
Sunday

Can you do this so that each table has 2 men and 2 women?

Must one avoid one's spouse??

No! You now have an excuse for another dinner!

Cycle systems of graphs other than K_n are also interesting.



Join vertices in the same group with λ_1 edges and vertices in different groups with λ_2 edges

$\lambda_1 = 2$ and $\lambda_2 = 1$

Pure and Mixed Edges

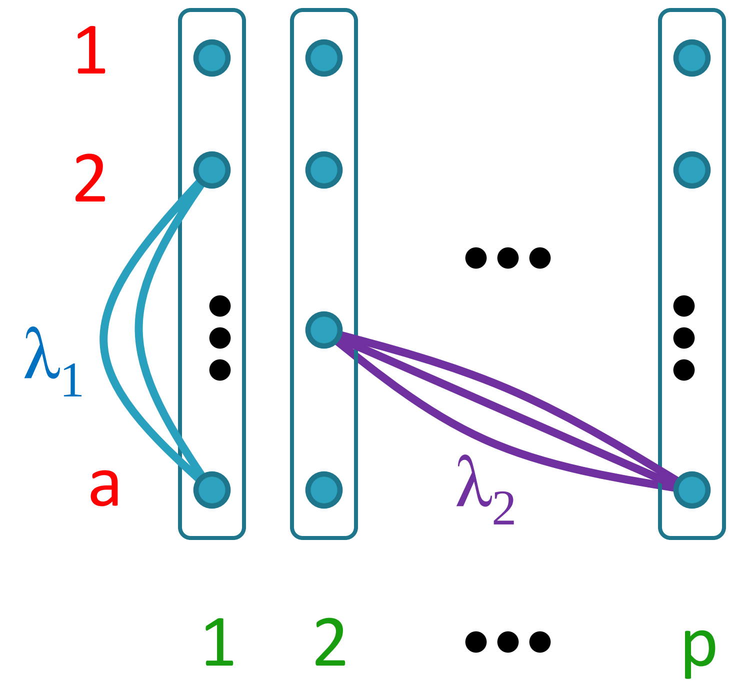
Cycle Systems with 2 Associate Classes

Maybe you have one big table!

There exists a C_{ap}-factorization of $K(a,p;\lambda_1,\lambda_2)$ if and only if:

1. $\lambda_1 (a-1) + \lambda_2 a(p-1)$ is even, and
2. $\lambda_2 a(p-1) \geq \lambda_1$.

(Bahmanian, Rodger)



$K(a,p;\lambda_1,\lambda_2)$

Cycle Systems with 2 Associate Classes

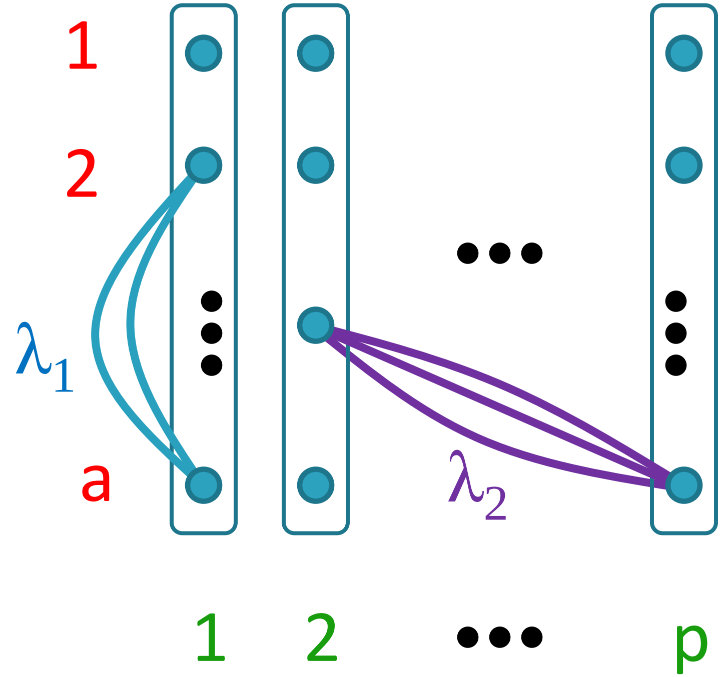
Tables of size 4 are more common!

Suppose a is even.

There exists a C_4-factorization of $K(a,p;\lambda_1,\lambda_2)$ if and only if

1. 4 divides ap
2. λ_1 is even, and
3. If $a \equiv 2 \pmod 4$ then $\lambda_2 a(p-1) \geq \lambda_1$.

(Billington, Rodger)



$K(a,p;\lambda_1,\lambda_2)$

Cycle Systems with 2 Associate Classes

Tables of size 4 are more common!

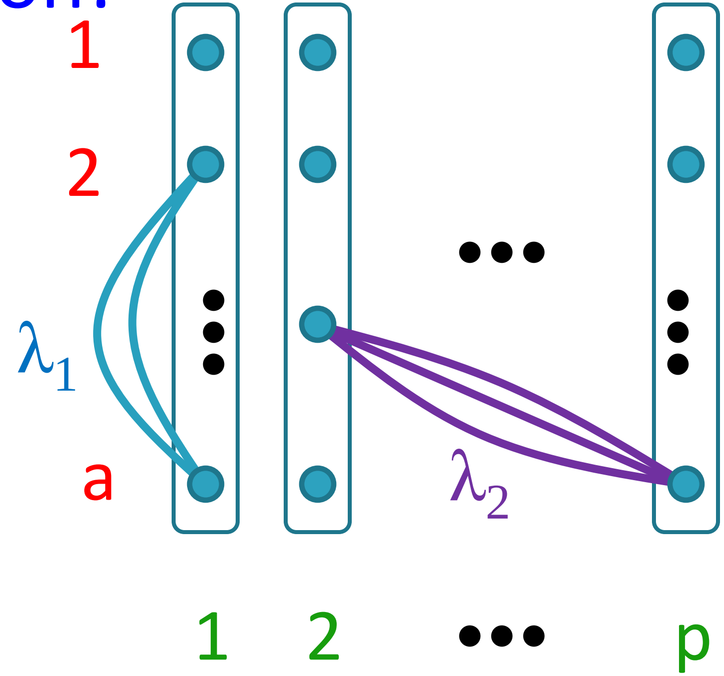
Suppose a $\equiv 1 \pmod 4$.

There exists a C_4-factorization of $K(a,p;\lambda_1,\lambda_2)$ if and only if

1. 4 divides p
2. $\lambda_2 > 0$ and is even, and
3. $\lambda_2 a(p-1) \geq \lambda_1$,

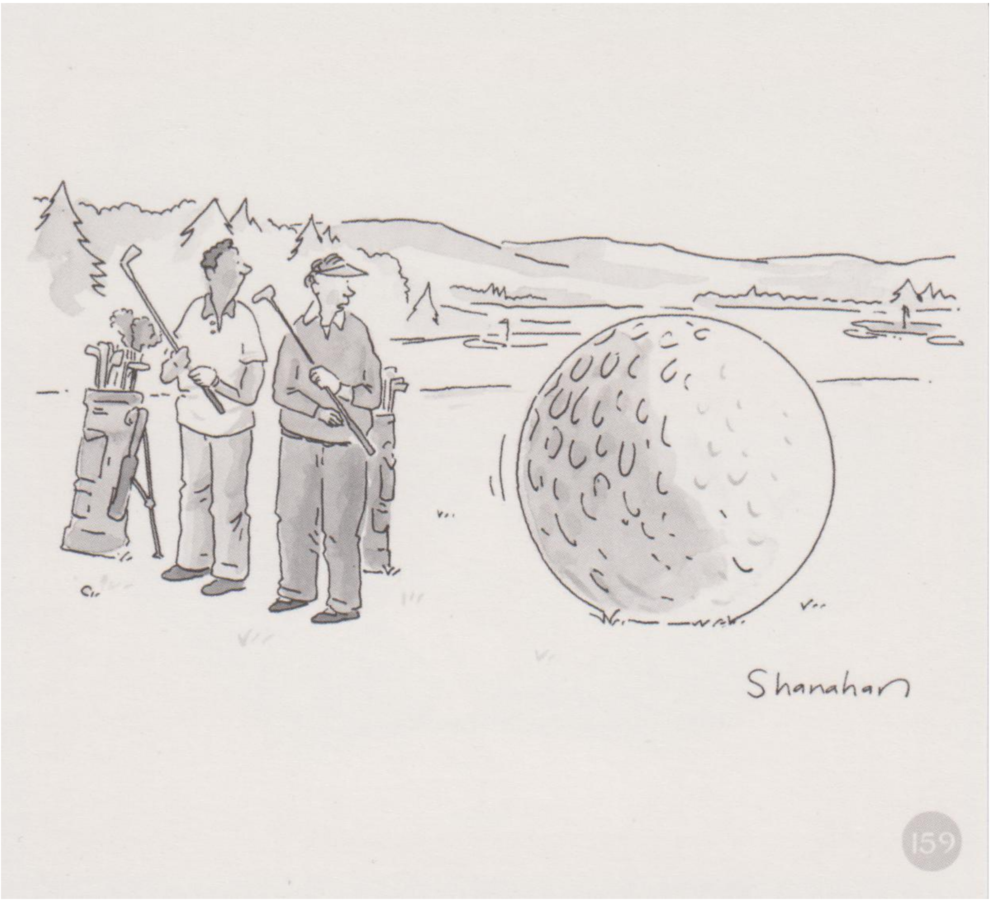
except possibly if $a = 9$ and λ_1 is odd.

(Rodger, Tiemeyer)



$K(a,p;\lambda_1,\lambda_2)$

What about a $\equiv$ 3 (mod 4)?



Looks difficult from my point of view!

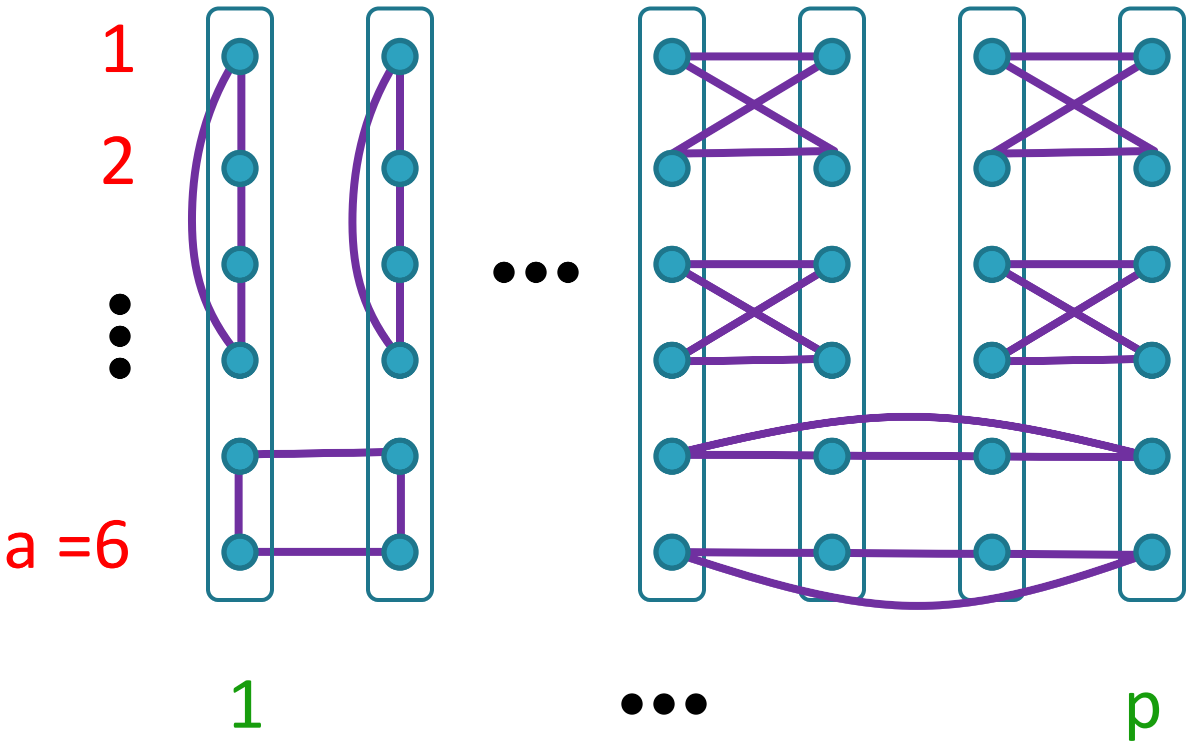
Why must $\lambda_2 a(p-1) \geq \lambda_1$?

Suppose a $\equiv 2 \pmod 4$.

Consider one C_4-factor.

Every part must contain at least 2 vertices incident with mixed edges.

So each C_4-factor must contain at least p mixed edges!

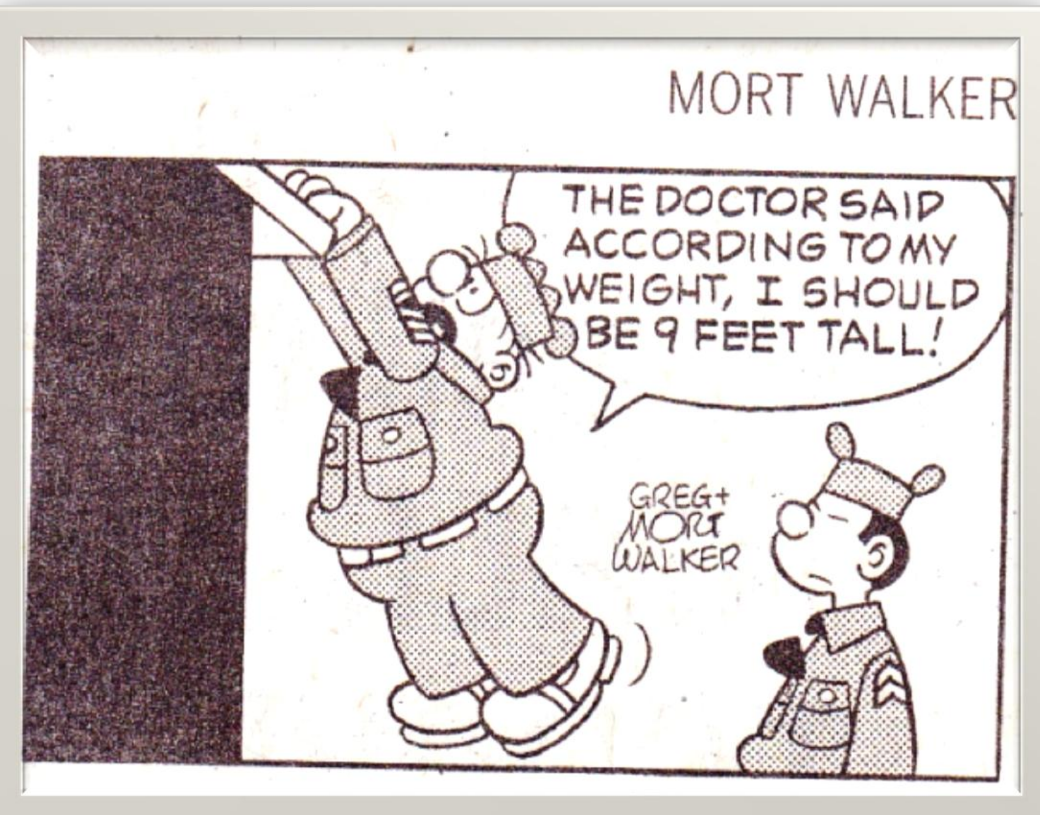


The same argument works for hamilton cycles.

$K(a,p;\lambda_1,\lambda_2)$

How do the proofs go?

You need a different perspective!



For the hamilton cycles, use amalgamations!

That approach also lets you prove embedding results!

4-cycle systems of $K(a,p;\lambda_1,\lambda_2)$

There exists a 4-cycle system of $K(a,p;\lambda_1,\lambda_2)$ if and only if

1. Each vertex has even degree,
2. The number of edges is divisible by 4,
3. If a = 2 then
   - $\lambda_2 > 0$, and
   - $\lambda_1 \leq 2(p-1)\lambda_2$
4. If a = 3 then
   - $\lambda_2 > 0$, and
   - $\lambda_1 \leq 3(p-1)\lambda_2/2$ if λ_2 is even, and
   - $\lambda_1 \leq 3(p-1)\lambda_2/2 - (p-1)/9$ if λ_2 is odd.

(Hung Lin Fu, Rodger)

For 3-cycles: Fu, Rodger, Sarvate

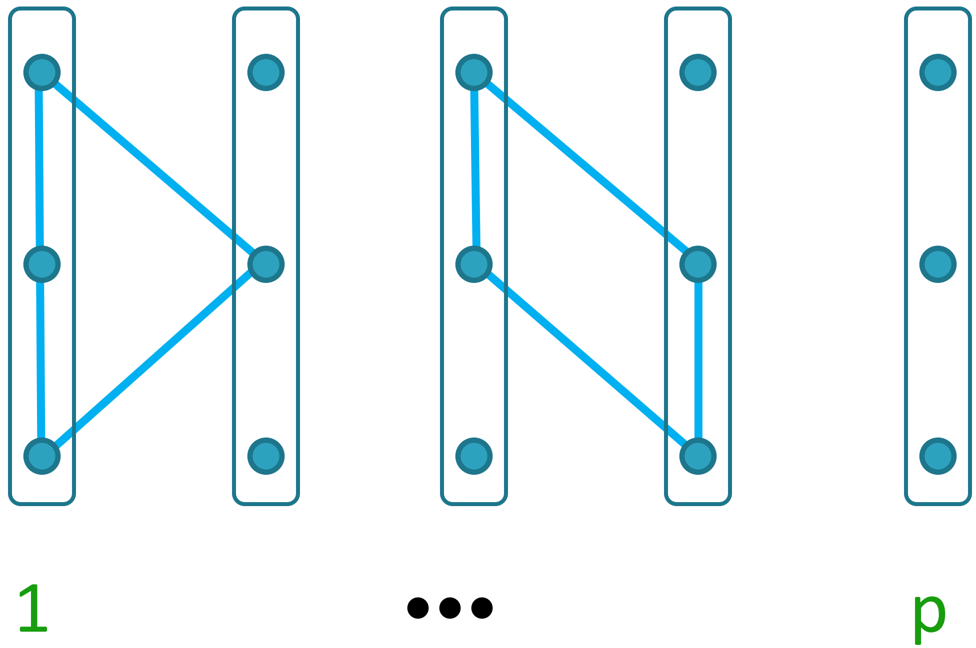
For block designs: Bose and Shimamoto – 1952!

Why is $\lambda_1 \leq 3(p\text{-}1)\,\lambda_2/2$ when $a = 3$?

- Every 4-cycle must use at least 2 mixed edges.
- So $3p\lambda_1 \leq 9\lambda_2 p(p\text{-}1)/2$

$K(a,p;\lambda_1,\lambda_2)$

Each of these uses an **even** number of mixed edges.

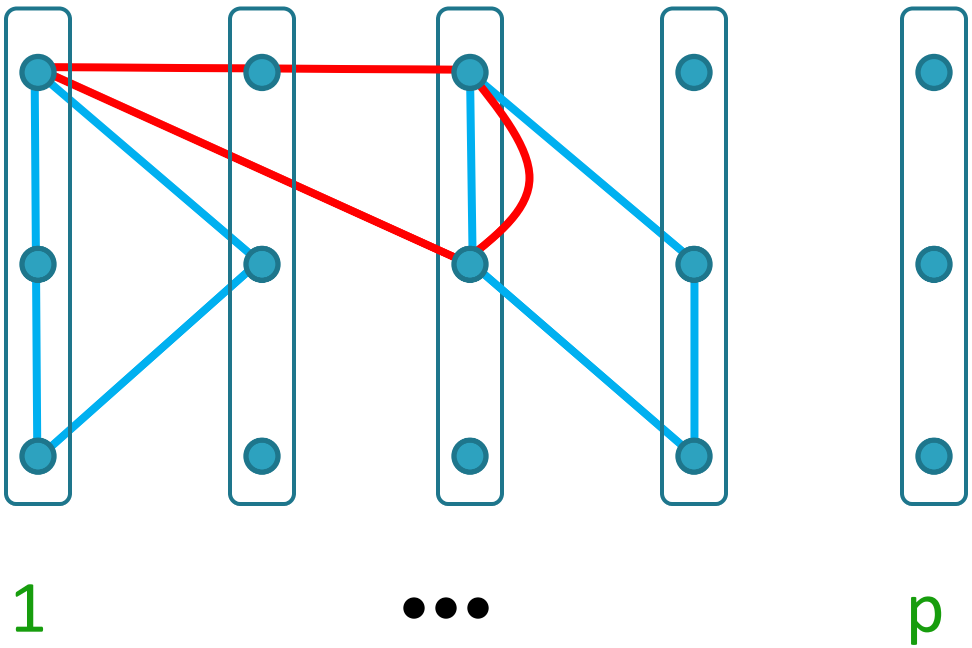


Is $\lambda_1 \le 3(p\text{-}1)\,\lambda_2/2 - (p\text{-}1)/9$ when λ_2 is odd?

- There are an *odd* number of edges between each pair of parts!
- So some 4-cycles must use at least 3 mixed edges
- So $3p\lambda_1 - p(p\text{-}1)/6 \le 9\lambda_2 p(p\text{-}1)/2 - p(p\text{-}1)/2$

$K(a,p;\lambda_1,\lambda_2)$

Each of these uses an **even** number of mixed edges.



Plenty More!

- Cycle systems that cover 2-paths (4-cycles)

  (Heinrich and Nonay, Cox and Rodger)
- Resolvable versions

  (Kobayashi and Nakamura)
- Fair and gregarious cycle systems of multipartite graphs
- Cycle systems of line graphs of K_n and of line graphs of complete multipartite graphs (4-cycles)

  (Rodger and Sehgal)
- Cycle systems (3- and 4-cycles) of K_n minus any graph with
  - maximum degree 3
  - One vertex of any degree, all others of degree at most 2

  (Fu, Fu and Rodger, Sehgal, Ash)

Thanks for listening!

Not quite time for a cuppa!!