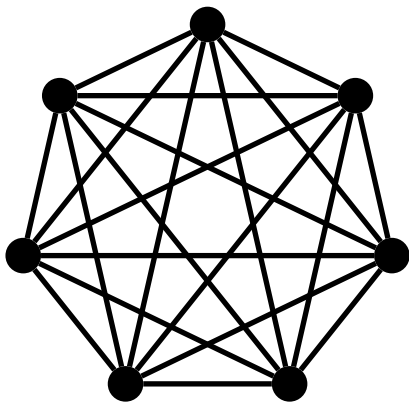


Embeddings of partial Steiner triple systems: best-possible and better

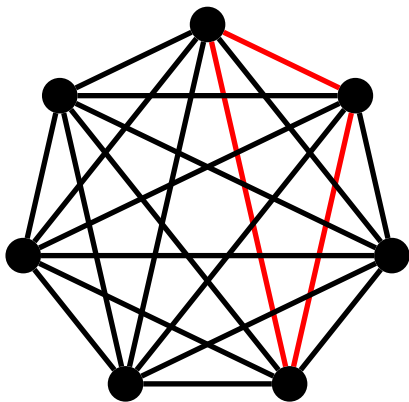
Daniel Horsley (Monash University)

Steiner triple systems

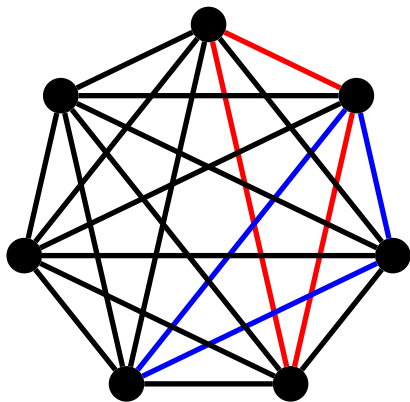
Steiner triple systems



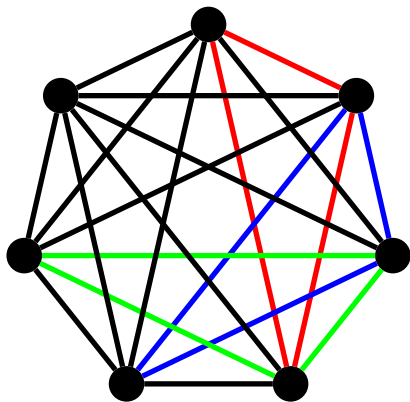
Steiner triple systems



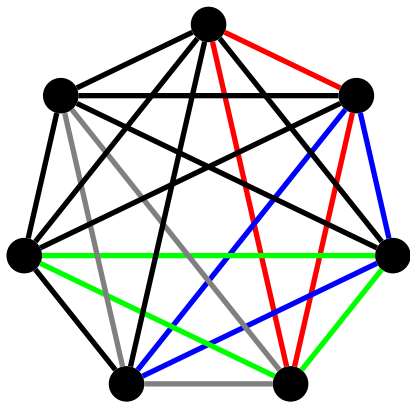
Steiner triple systems



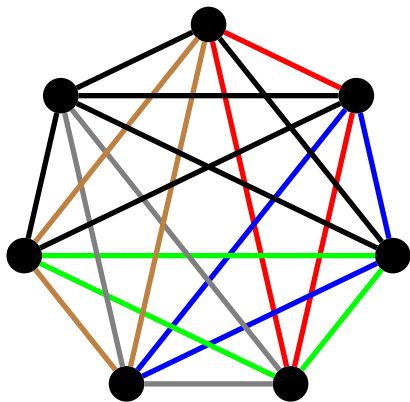
Steiner triple systems



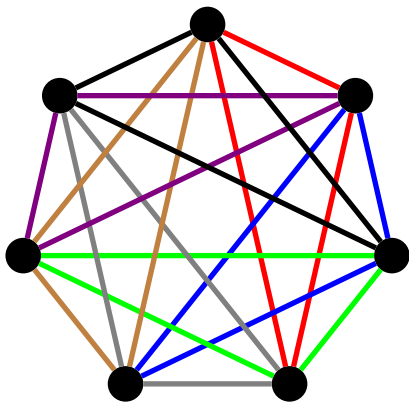
Steiner triple systems



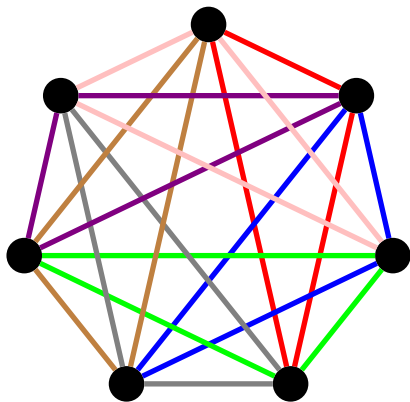
Steiner triple systems



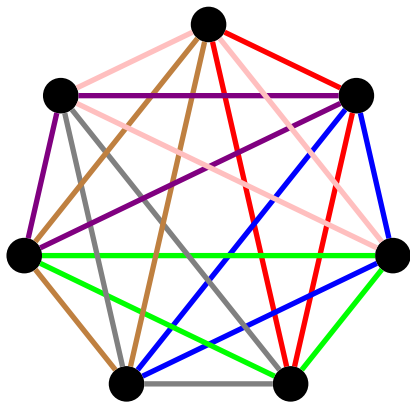
Steiner triple systems



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An $STS(7)$

Existence of Steiner triple systems

Existence of Steiner triple systems

Theorem (Kirkman 1847) An STS(v) exists if and only if $v \geq 1$ and $v \equiv 1, 3 \pmod{6}$.

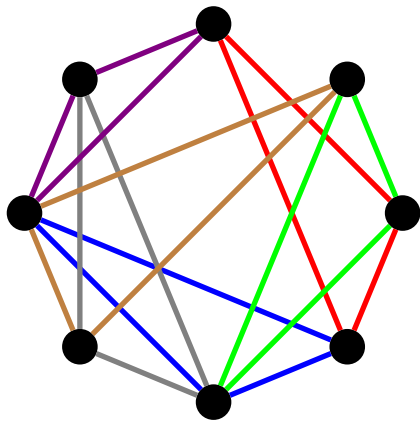
Existence of Steiner triple systems

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Call these integers *admissible*.

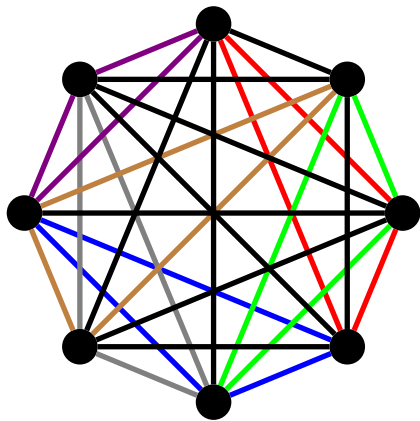
Partial Steiner triple systems

Partial Steiner triple systems



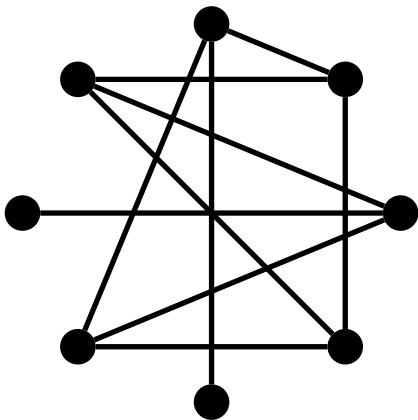
A PSTS(8)

Partial Steiner triple systems



A PSTS(8)

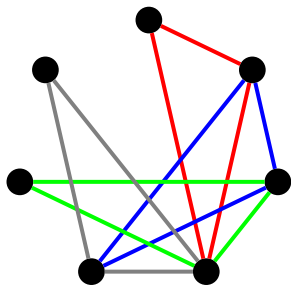
Partial Steiner triple systems



The leave of the PSTS(8)

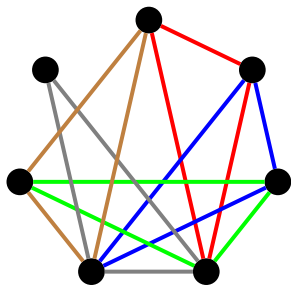
Completing partial Steiner triple systems

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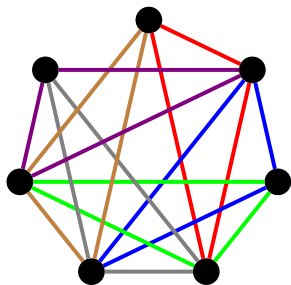
A PSTS(7)

Completing partial Steiner triple systems



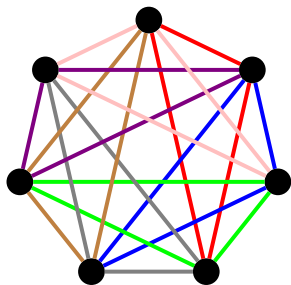
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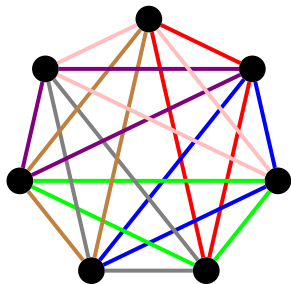
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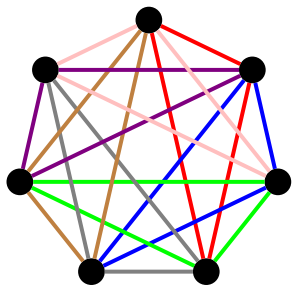
A $\text{PSTS}(7)$

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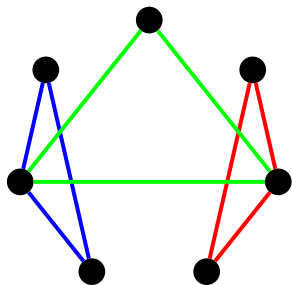


A completion of the $\text{PSTS}(7)$

Completing partial Steiner triple systems

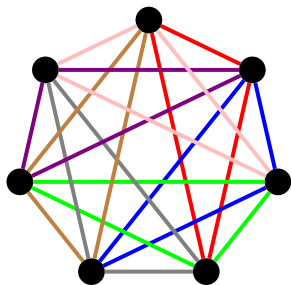


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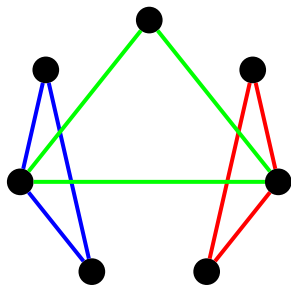


A $\text{PSTS}(7)$

Completing partial Steiner triple systems



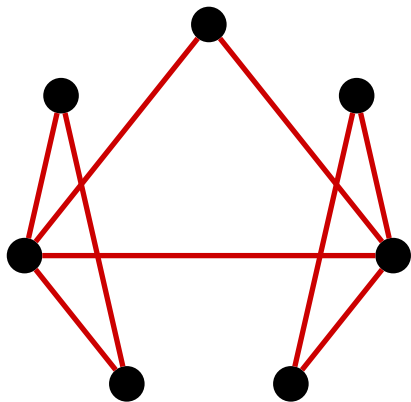
A completion of the PSTS(7)



A PSTS(7) with no completion

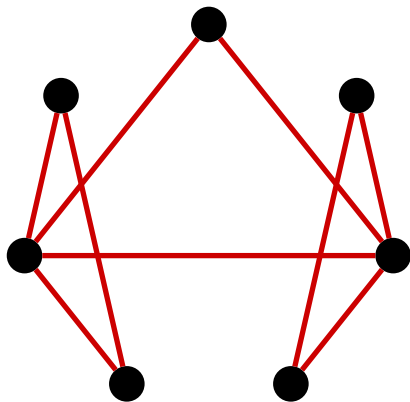
Embeddings of partial Steiner triple systems

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A PSTS(7)

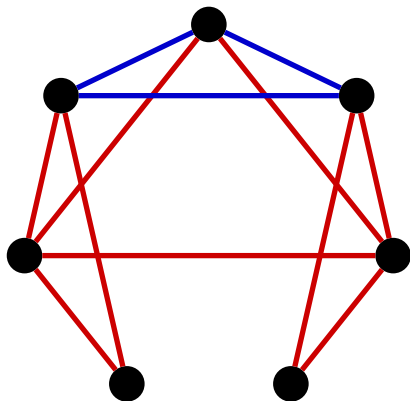
Embeddings of partial Steiner triple systems



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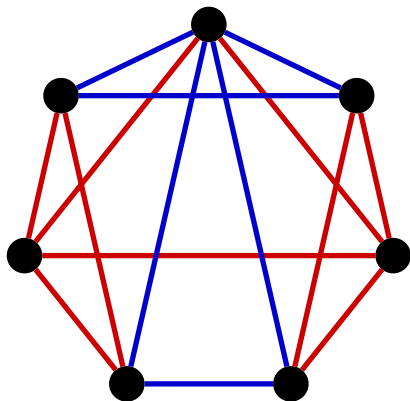


Embeddings of partial Steiner triple systems



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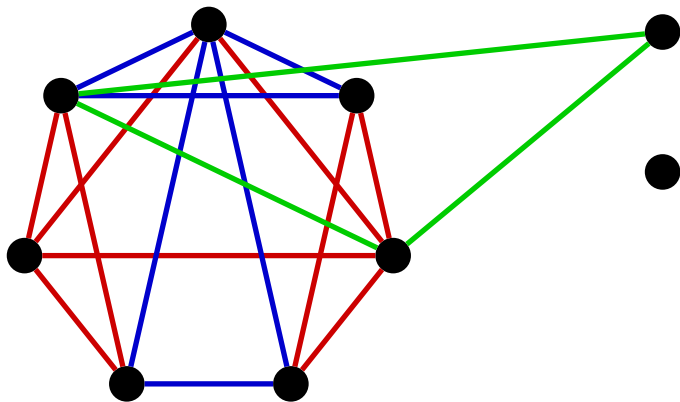
Embeddings of partial Steiner triple systems



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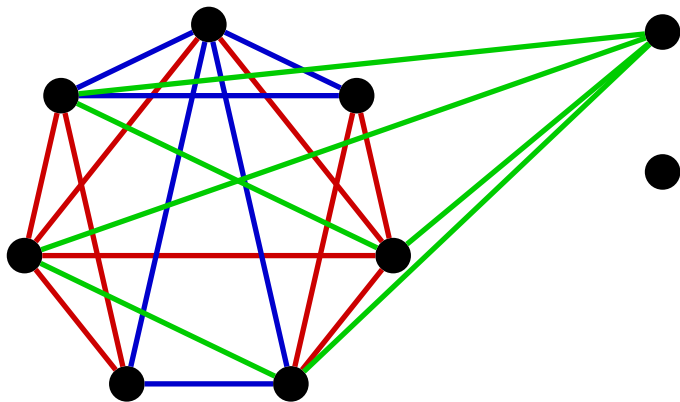


Embeddings of partial Steiner triple systems



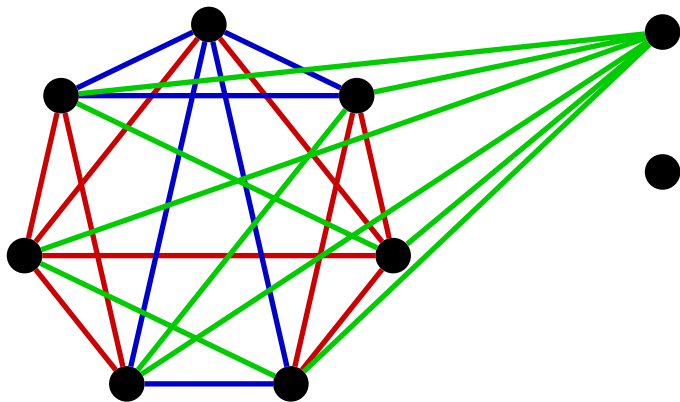
A PSTS(7)

Embeddings of partial Steiner triple systems



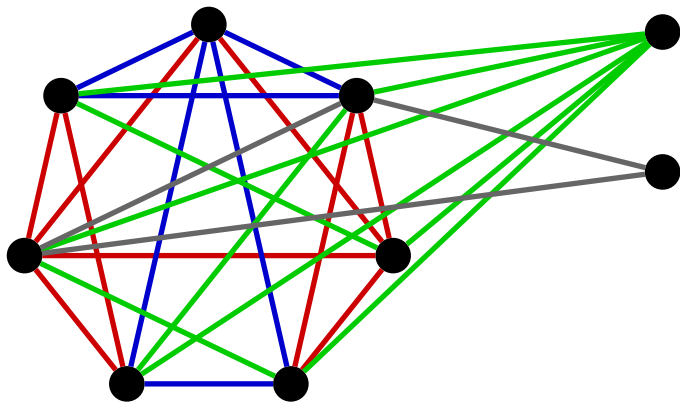
A PSTS(7)

Embeddings of partial Steiner triple systems



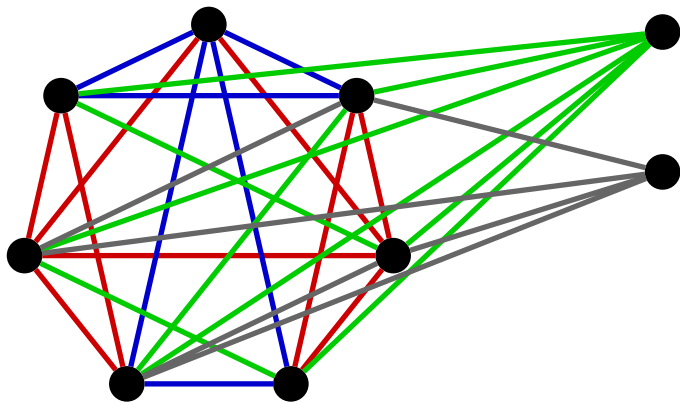
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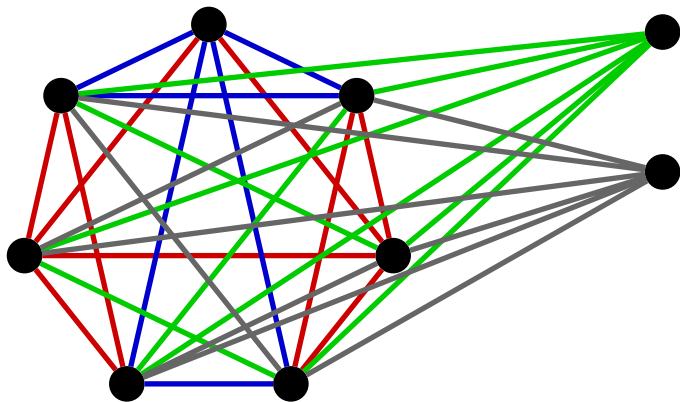
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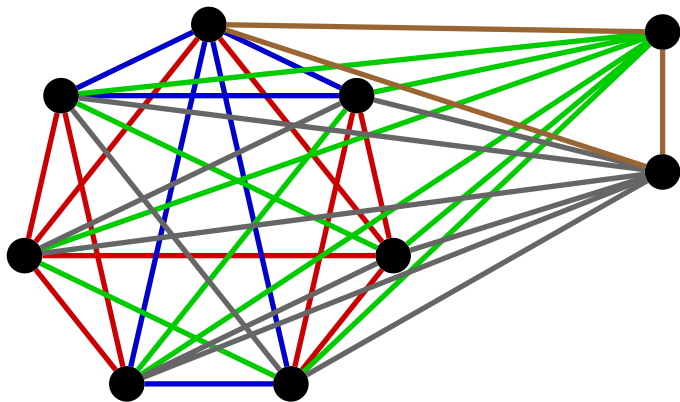
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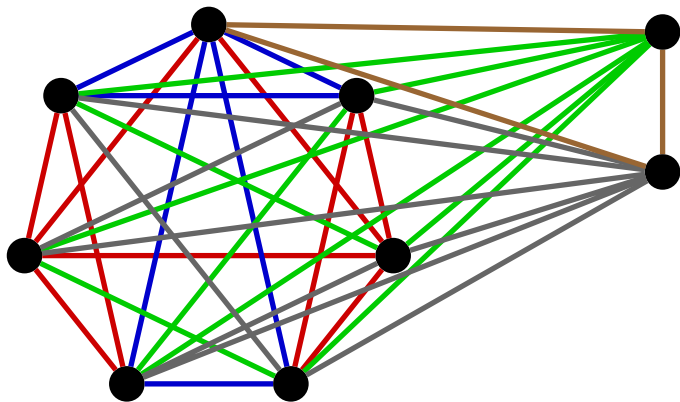
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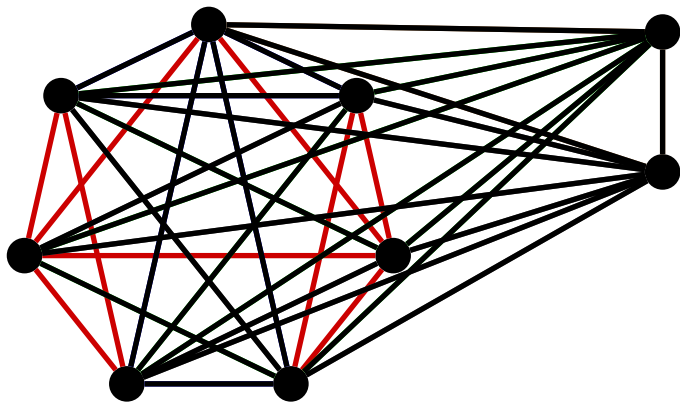
A PSTS(7)

Embeddings of partial Steiner triple systems



An embedding of the PSTS(7) of order 9

Embeddings of partial Steiner triple systems



$L \vee K_2$ was decomposed into triangles

The blind problem

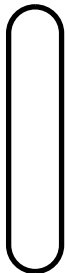
The blind problem

Problem For what values of v does *every* $\text{PSTS}(u)$ have an embedding of order v ?

Small embeddings may not exist

Small embeddings may not exist

u vertices



$|E(L)|$

Small embeddings may not exist

u vertices



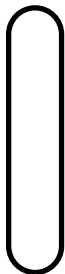
$|E(L)|$

w vertices



Small embeddings may not exist

u vertices



$|E(L)|$

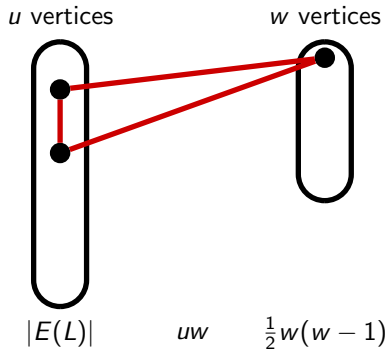
w vertices



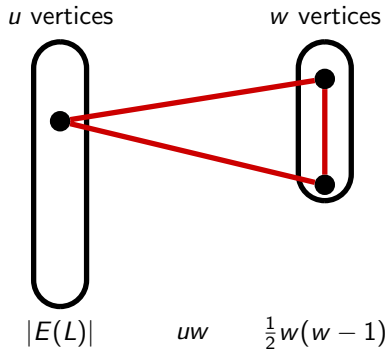
uw

$\frac{1}{2}w(w-1)$

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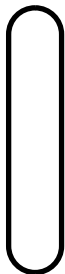


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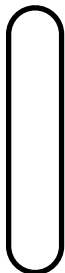


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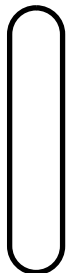
uw

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$$uw \leq 2(|E(L)| + \frac{1}{2}w(w-1))$$

Small embeddings may not exist

u vertices



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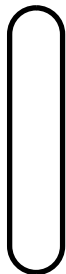
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$\frac{1}{2}w(w-1)$

$$uw \leq 2(|E(L)| + \frac{1}{2}w(w-1)) \Rightarrow |E(L)| \geq \frac{1}{2}w(u-w+1)$$

Small embeddings may not exist

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uw

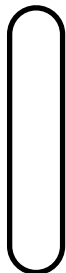
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If $|E(L)| = 6$, then we must have $w = 0$ or $w \geq u + 1$ (for decent size u).

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Result For each $u > 9$ there is a PSTS(u) which does not have any embedding of order smaller than $2u + 1$.

Results on the blind problem

Treash (1971) Every PSTS has an embedding (of order at most 2^{2u}).

Lindner (1975) Every $\text{PSTS}(u)$ has an embedding of order $6u + 3$.

Conjecture (Lindner 1977) Every $\text{PSTS}(u)$ has an embedding of order v for each admissible $v \geq 2u + 1$.

Andersen, Hilton, Mendelsohn (1980) Every $\text{PSTS}(u)$ has an embedding of order v for each admissible $v \geq 4u + 1$.

Bryant (2004) Every $\text{PSTS}(u)$ has an embedding of order v for each admissible $v \geq 3u - 2$.

Bryant, Horsley (2009) Every $\text{PSTS}(u)$ has an embedding of order v for each admissible $v \geq 2u + 1$.

The sighted problem

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Call embeddings of order less than $2u + 1$ *small*.

Some necessary conditions

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Result Suppose there is an embedding of a PSTS(u) of order $u + w$. If L is the leave of the PSTS, then there is a subgraph L' of L such that

- ▶ $L - L'$ has a decomposition into triangles;
- ▶ $\chi'(L') \leq w$; and
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Some necessary conditions

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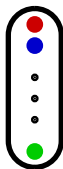
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w vertices

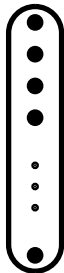


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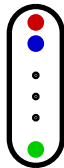
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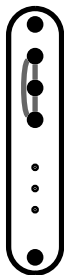


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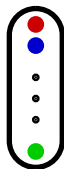
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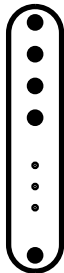


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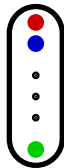
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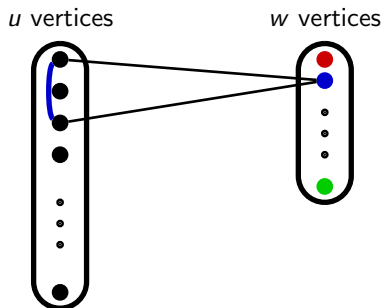
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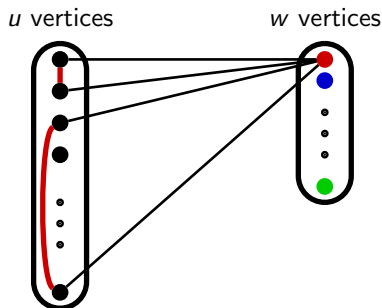
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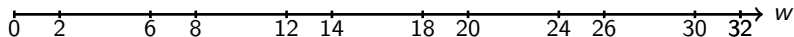
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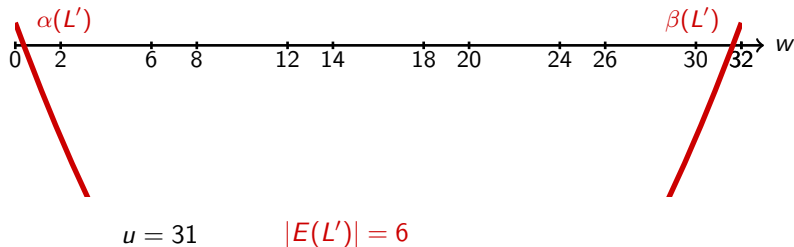


$$u = 31$$

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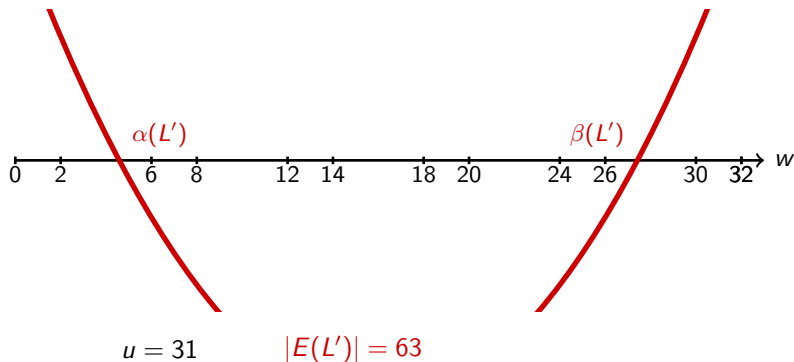
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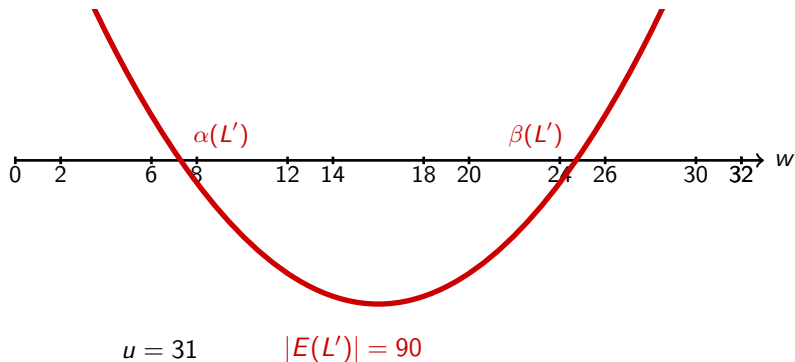
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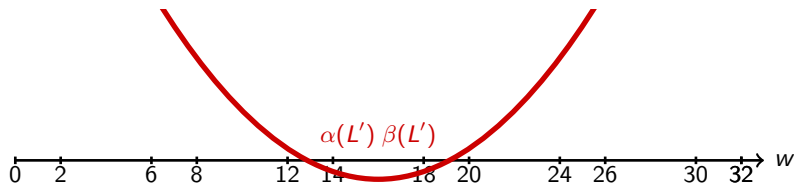
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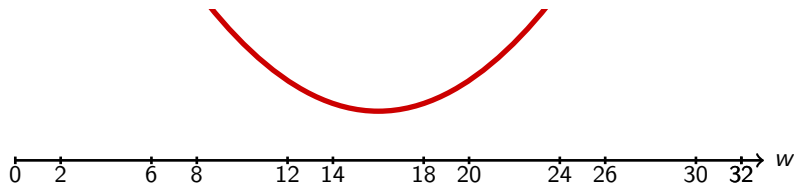
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A closer look at those conditions

Result Suppose there is an embedding of a PSTS(u) of order $u + w$. If L is the leave of the PSTS, then there is a subgraph L' of L such that

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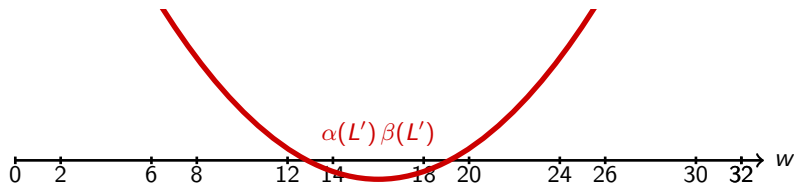
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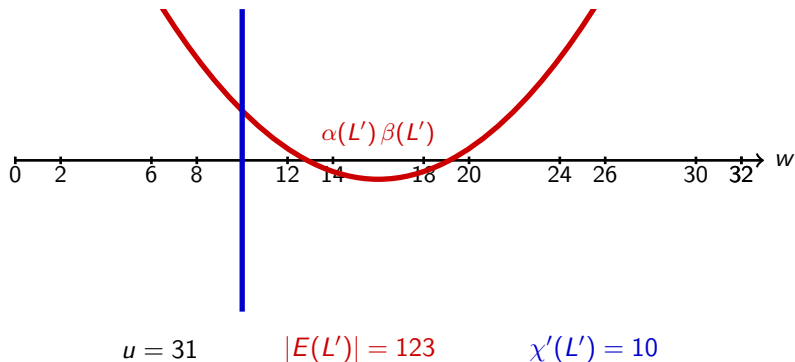
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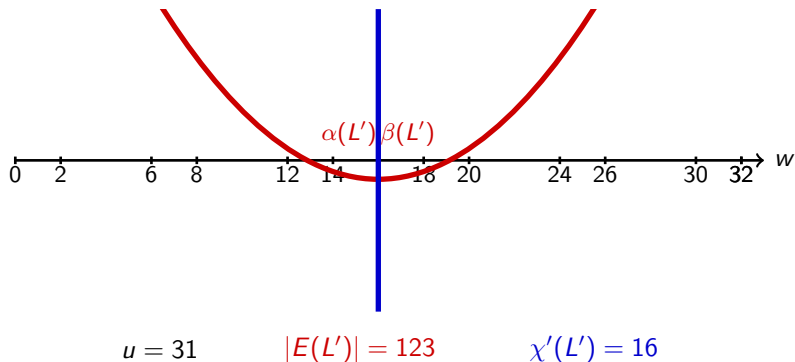
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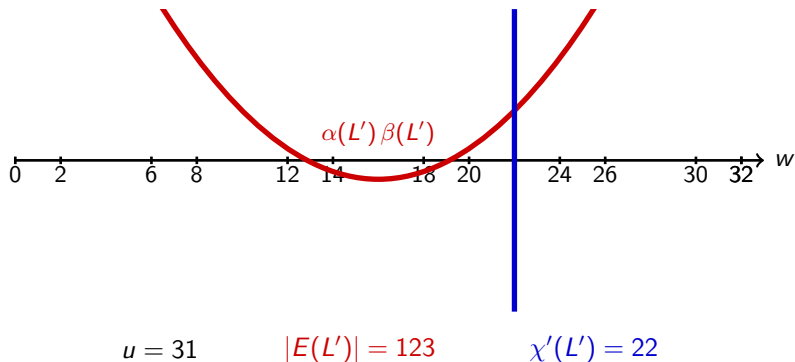
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Finding embeddings is NP-complete

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Colbourn (1983) constructed a class C of PSTSs such that

- ▶ it is NP-complete to determine whether a member of C has a completion;
- ▶ no member of C can be embedded in an $\text{STS}(v)$ for any $u < v < 2u + 1$.

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- ▶ it is NP-complete to determine whether a member of C has a completion;
- ▶ no member of C can be embedded in an STS(v) for any $u < v < 2u + 1$.

This means that determining whether a PSTS has an embedding of order less than $2u + 1$ is NP-complete.

Results on the sighted problem

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Doyen, Wilson (1973) A (complete) STS(u) has an embedding of order v if and only if $v = u$ or $v \geq 2u + 1$.

Colbourn, Colbourn, Rosa (1983) Showed “double stars” are completable.

Bryant (2002) Gave necessary and sufficient conditions for a PSTS with a $\{0, d\}$ -regular leave to have an embedding of order $u + d$. Determined the embedding spectrum in the case $d = 2$.

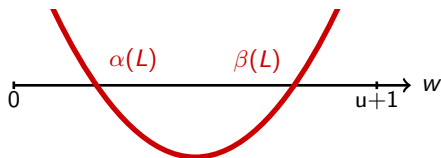
Bryant, Maenhaut, Quinn, Webb (2004) Determined the embedding spectrum for all PSTS(10)s with 3-regular leaves.

Bryant, Horsley (2006) Determined the embedding spectrum for PSTSs with complete bipartite leaves.

A new result

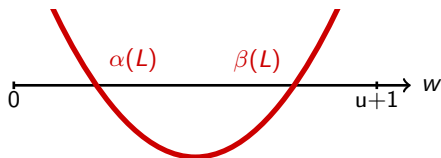
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Theorem Any PSTS(u) with a leave L such that $\Delta(L) \leq \frac{1}{4}(u - 9)$ and $|E(L)| < \frac{1}{32}(u - 5)(u - 11) + 2$ has an embedding of order $u + w$ for each integer w such that $w \geq \beta(L)$ and $u + w$ is admissible.



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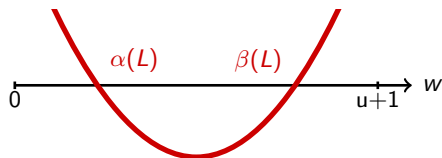
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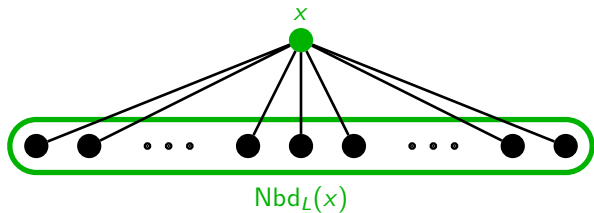
When $|E(L)| \approx \frac{1}{32}(u - 5)(u - 11) + 2$, we have

$$\alpha(L) \approx \frac{1}{4}(2 - \sqrt{3})(u + 1) \approx 0.07(u + 1),$$

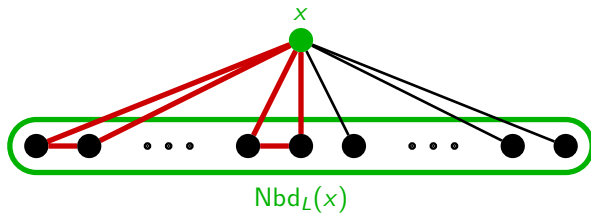
$$\beta(L) \approx \frac{1}{4}(2 + \sqrt{3})(u + 1) \approx 0.93(u + 1).$$

Finding embedding spectra: neighbourhoods

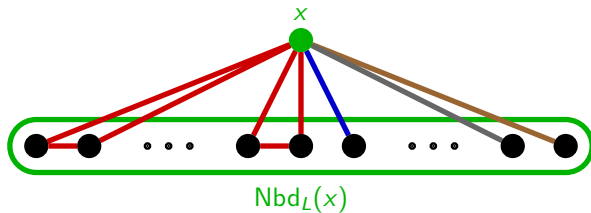
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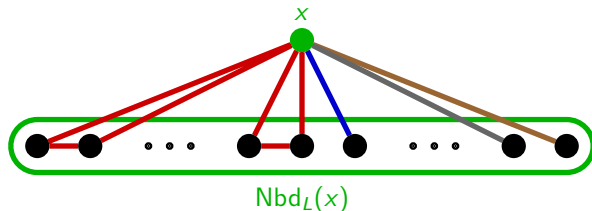
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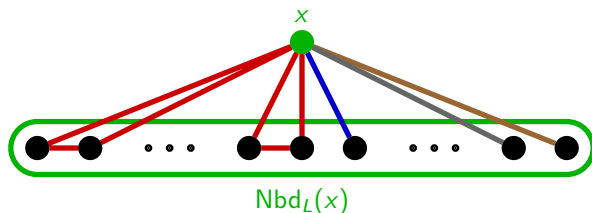


Finding embedding spectra: neighbourhoods



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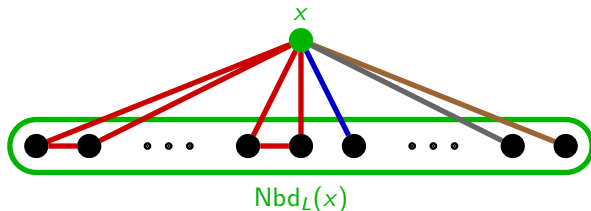
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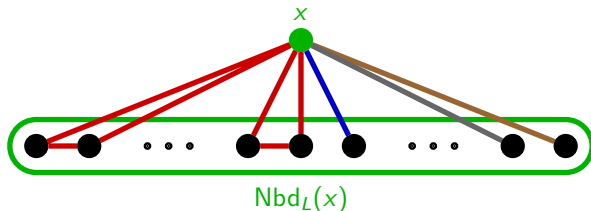
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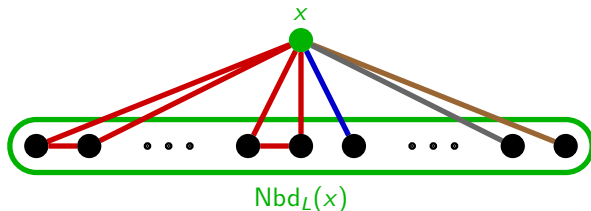
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Lemma A PSTS(u) has no embeddings of order at most $u + \alpha(L)$, except perhaps for a completion, if $|E(L)| < \frac{1}{2}(a+1)(u-a)$, where a is the number of isolated vertices in L .

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A more general result covering leaves with many vertices of small degree can be obtained.

Final thoughts

- ▶ isolating NP-completeness
- ▶ partially sighted questions

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Not a theorem yet Any PSTS(u) with $u \geq 57$ and at most $\frac{1}{60}u(u-1)$ triples has an embedding of order $u+w$ for each integer w such that $w \geq \frac{1}{5}(3u+17)$.

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There is a PSTS(u) with about $\frac{1}{60}u(u-1)$ triples which does not have any embedding of order $u+w$ for $w \leq 0.31(u+1)$.

The End