Graph Symmetries

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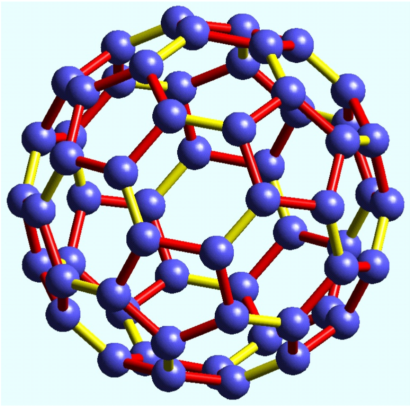
Talk dedicated to our late colleague
Peter Lorimer

What is symmetry?

Symmetry can mean many different things, such as balance, uniform proportion, harmony, or congruence

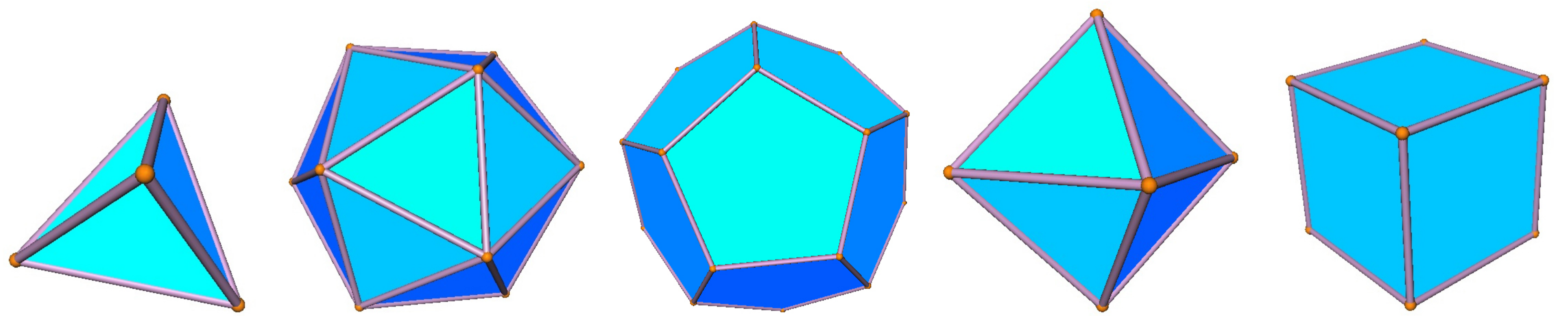
  

Generally, an object has symmetry if it can be transformed in way that leaves it looking the same as it did originally.

Examples of symmetric structures abound in nature

... but have also been manufactured by human fascination and enterprise

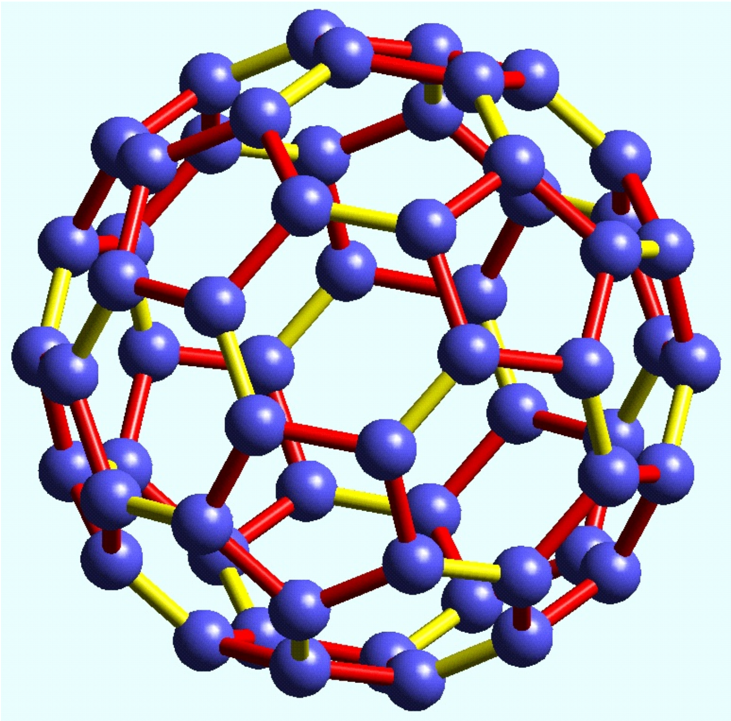
e.g. the Platonic solids (c. 360BC)



Tetrahedron Icosahedron Dodecahedron Octahedron Cube

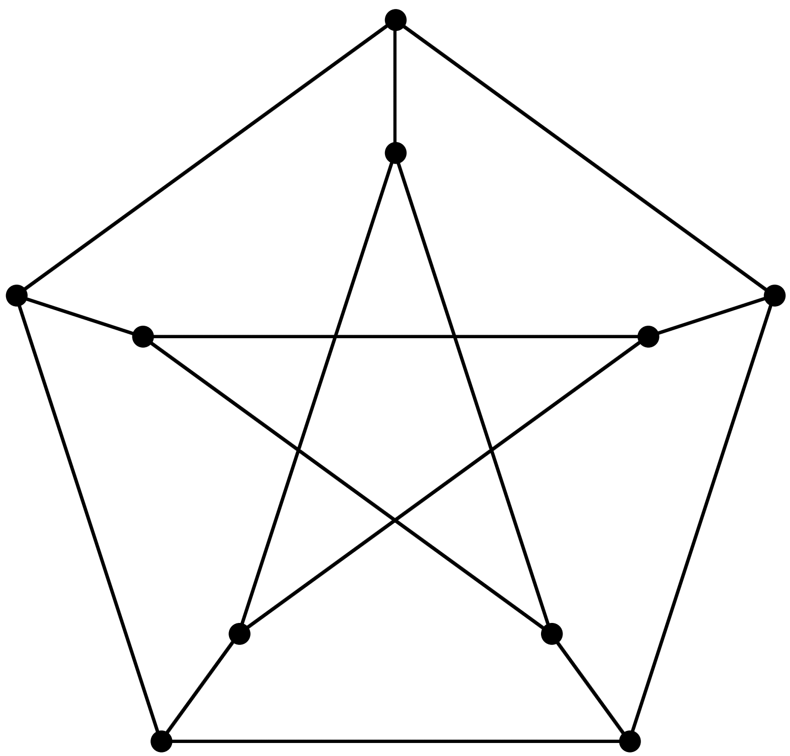
Symmetry can induce strength and stability:

... or its more contemporary version, the C_{60} molecule **Buckminsterfullerene** (“buckyball”):

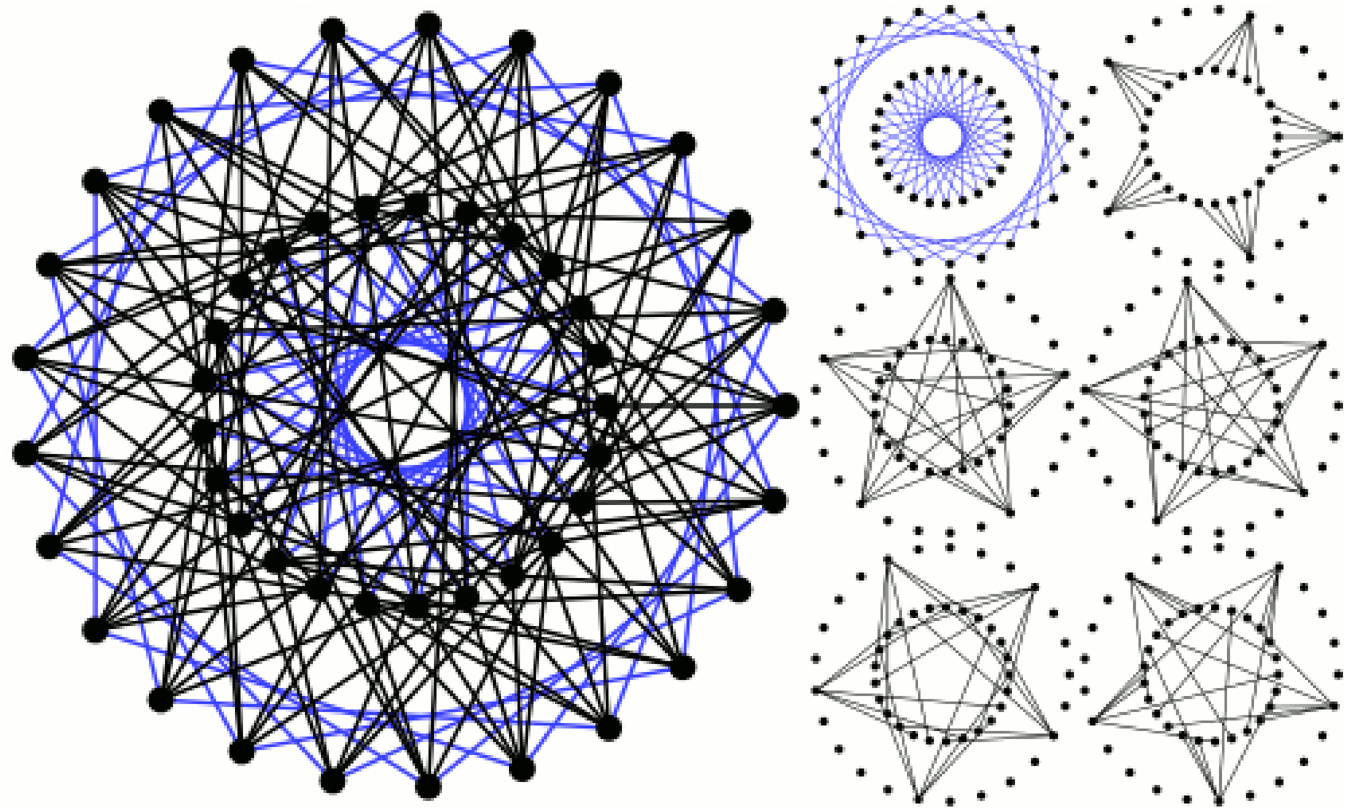


Symmetry can also arise unexpectedly ...

The largest connected graph with (maximum) degree 3 and diameter 2 is the Petersen graph:



The largest 7-valent graph of diameter 2



... is also highly symmetric: the Hoffman-Singleton graph

Graph symmetries

A symmetry (or automorphism) of a simple graph X is a permutation of its vertices preserving adjacency.

(For a multigraph $X = (V, E)$, we can take this as a permutation of $V \cup E$ preserving V and E and preserving incidence.)

In particular, every automorphism preserves valency: if the vertex v has degree/valency k, then so does the image of v under every automorphism.

Also every automorphism preserves other things, such as the lengths of cycles containing the vertex.

Under composition, the symmetries of a given graph X form a group Aut(X), called the automorphism group of X.

Examples: Automorphism groups of graphs

Graph	Automorphism group
C_n (cycle), $n \geq 3$	D_n (order $2n$)
P_n (path), $n \geq 2$	C_2 (order 2)
K_n (complete), $n \geq 1$	S_n (order $n!$)
N_n (null), $n \geq 1$	S_n (order $n!$)
$K_{m,n}$ (complete bipartite), $m, n \geq 1$	$S_m \times S_n$ ($m \neq n$) or $S_m \wr C_2$ ($m = n$)
Petersen	S_5

Amazing fact: Frucht's theorem

In 1939, Robert(o) Frucht proved this theorem:

Given any finite group G, there exist infinitely many connected graphs X such that $\mathrm{Aut}(X)$ is isomorphic to G.

And then later, in 1949, he proved that X may be chosen to be 3-valent.

There are several variants and generalisations of this — e.g. regular representations for graphs and digraphs (GRRs and DRRs).

Some questions (for you to think about)

What are

- the smallest graph with exactly 3 symmetries?
- the connected 3-valent graphs of order 4, 6, 8 and 10 with the fewest symmetries?
- the smallest 3-valent graph with just one symmetry?
- the connected 3-valent graph of order 16 with the most symmetries?

Symmetry types

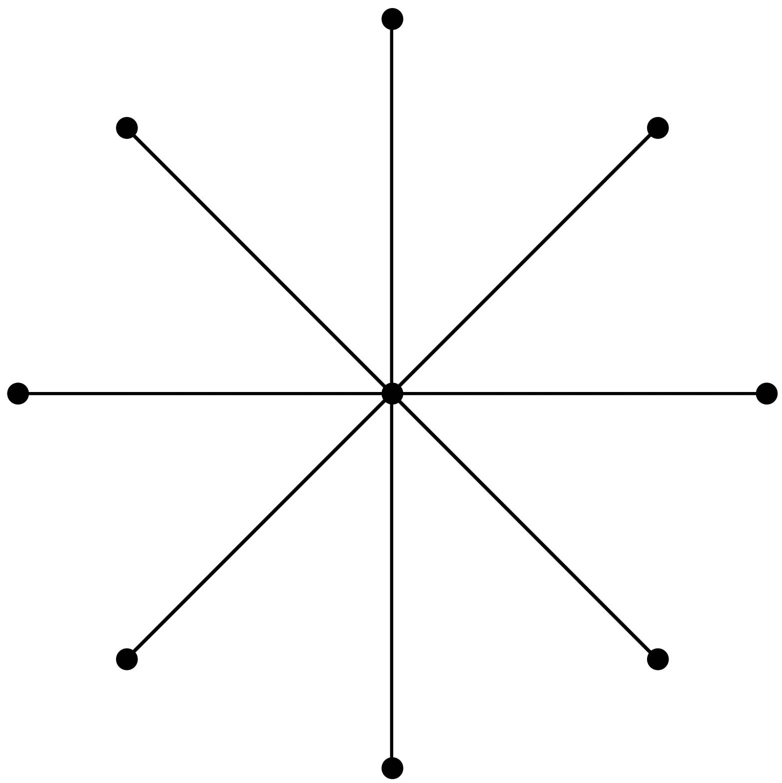
A graph is vertex-transitive if its automorphism group it transitive (i.e. has a single orbit) on the vertex set

A graph is edge-transitive if its automorphism group is transitive on the edge set

A graph is arc-transitive (also called symmetric) if its automorphism group is transitive on the set of all *arcs* (ordered pairs of adjacent vertices)

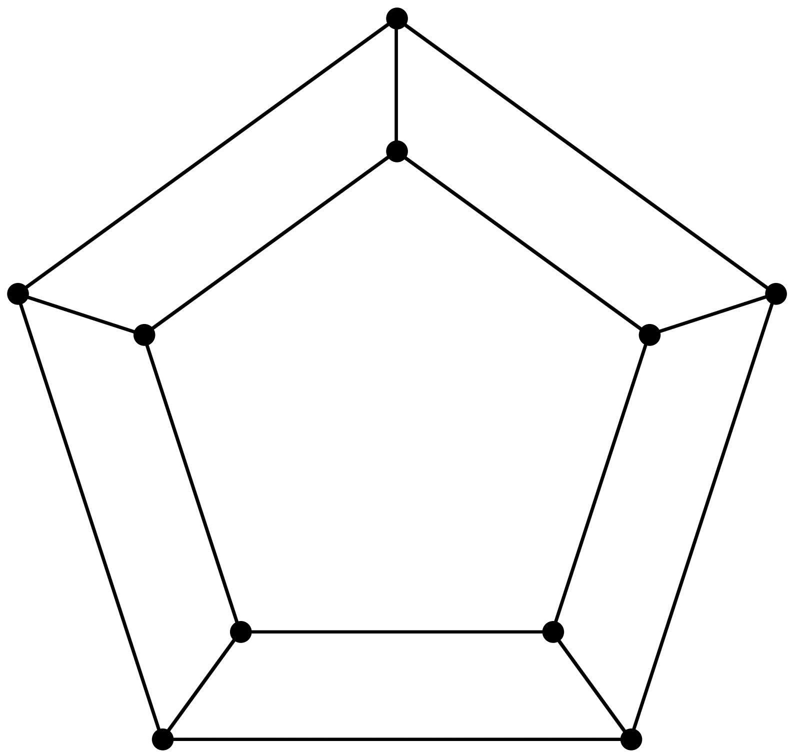
A graph is distance-transitive if for every positive integer d the automorphism group is transitive on the set of all order pairs of vertices (v, w) at distance d from each other.

Example:



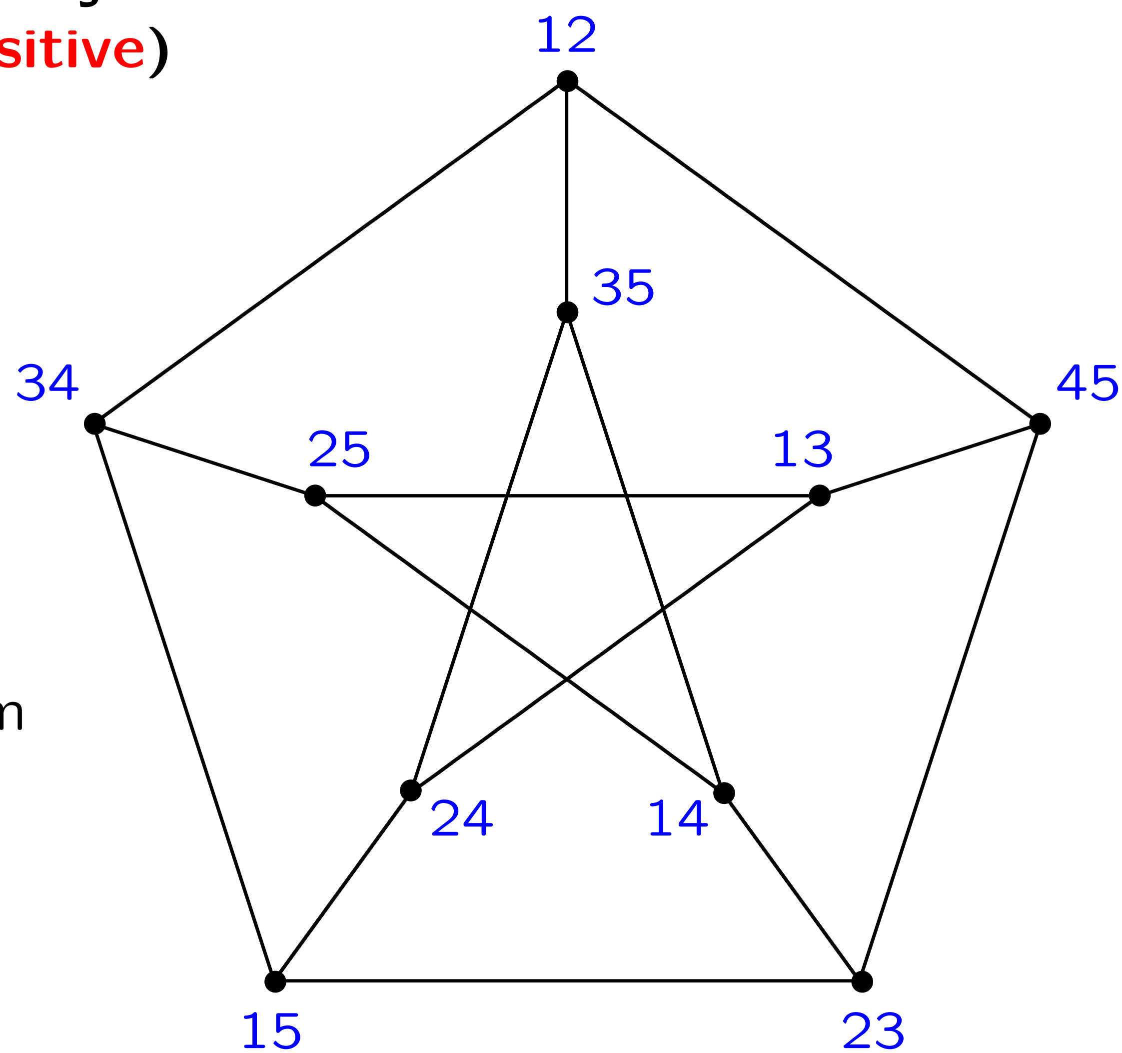
This (star graph) is edge-transitive but not vertex-transitive

Example:



This (prism) is vertex-transitive but not edge-transitive

The Petersen graph is symmetric
(in fact distance-transitive)



Every 3-path has form

ab — cd — ae — bc

Other symmetry types

A graph is half-arc-transitive if it is vertex-transitive and edge-transitive but not arc-transitive. The smallest example is 4-valent of order 28 ... constructed by Derek Holt (1981).

A graph is semi-symmetric if it is edge-transitive but not vertex-transitive. Any such graph is bipartite, with its parts being the two orbits of the automorphism group on vertices. The smallest regular example is a 4-valent graph of order 20 constructed by Jon Folkman (1967).

[And there are further generalisations of these]

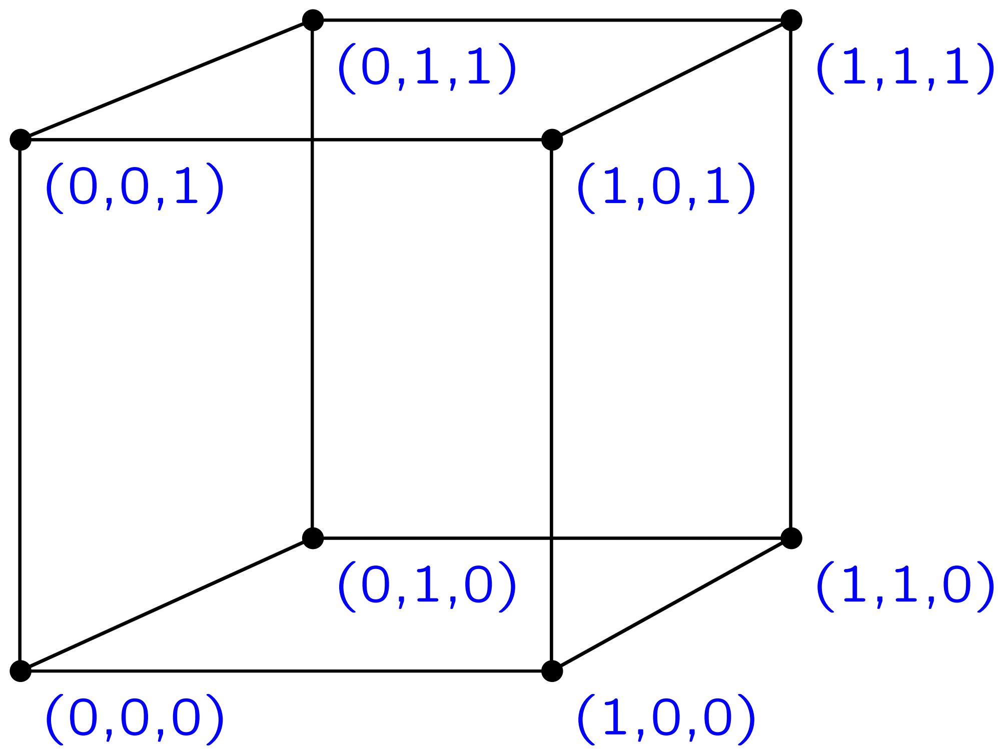
A construction for VT graphs: Cayley graphs

If X is a vertex-transitive graph, then $\mathrm{Aut}(X)$ acts transitively on $V(X)$. Conversely, given any group G, we can construct a graph X on which G acts as a vertex-transitive group of automorphisms.

Take $V(X) = G$ (so vertices of X are the elements of G), and for some generating set S for G, let $E(X)$ consist of all pairs of the form $\{v, xv\}$ where $v \in G$ and $x \in S$. This is the Cayley graph for G with respect to S, denoted by $\mathrm{Cay}(G, S)$.

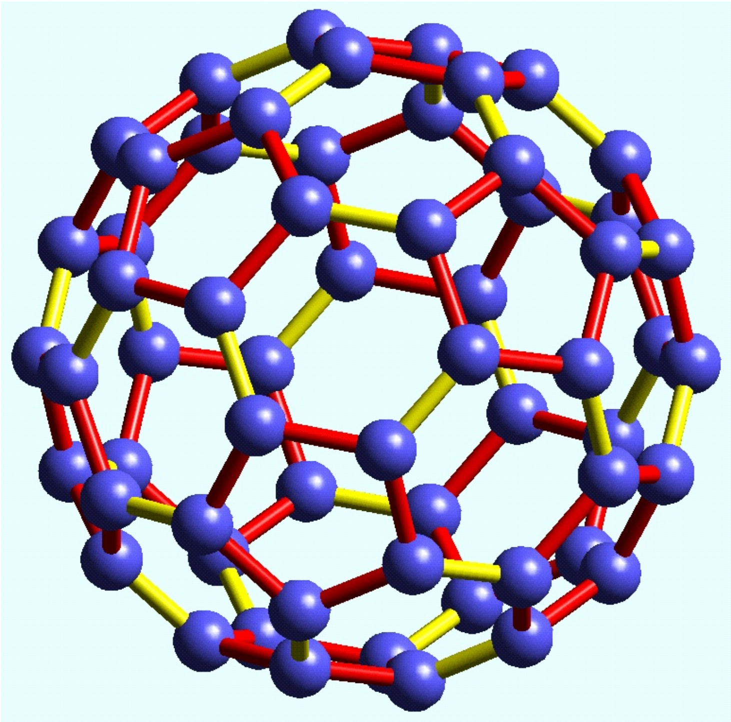
NB: G acts as a group of automorphisms of $X = \mathrm{Cay}(G, S)$, by right multiplication: $g \in G$ takes $\{v, xv\}$ to $\{vg, xvg\}$. And as vg can be any element, $X = \mathrm{Cay}(G, S)$ is vertex-transitive.

Example: the cube graph Q_3



... a Cayley graph for $\mathbb{Z}_2^3$ with $S = \{(1,0,0),(0,1,0),(0,0,1)\}$

Example: the buckminsterfullerene



… a Cayley graph for A_5 with $S = \{\underline{(1,2)(4,5)}, \underline{(1,2,3,4,5)}\}$

Double coset graphs (Sabidussi construction)

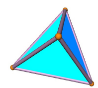
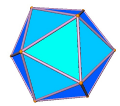
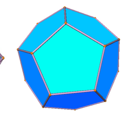
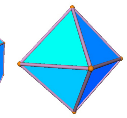
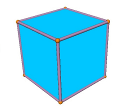
Let G be a group, H a subgroup of G, and a an element of G such that $a^2 \in H$. Now define a graph $X = X(G, H, a)$ by

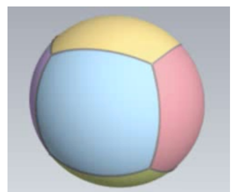
$$V(X) = \{Hg : g \in G\}$$
$$E(X) = \{\{Hx, Hy\} : x, y \in G \mid xy^{-1} \in HaH\}.$$

Then G induces a group of automorphisms of X by right multiplication. This action is vertex-transitive with vertex-stabiliser $G_H = \{g \in G : Hg = H\} = H$ itself, which acts transitively on the neighbours Hah (for $h \in H$) of H.
Thus $X(G, H, a)$ is arc-transitive!

– e.g. the Petersen graph is a double coset graph for A_5 (or S_5), but is not a Cayley graph.

Brief digression: symmetries of maps

The five Platonic solids      can be viewed as symmetric embeddings of graphs on the sphere

... e.g. the cube  [and video of other examples].

An automorphism of a map M is a permutation of the edges preserving incidence (with vertices and faces).

Every automorphism is uniquely determined by its effect on any incident vertex-edge-face triple, so $|\mathrm{Aut}(M)| \leq 4|E|$.

If the upper bound is attained, then M is a regular map.

Higher levels of symmetry

An s-arc in a graph X is a sequence $(v_0, v_1, v_2, \ldots, v_s)$ of $s+1$ vertices of X such that $\{v_{i-1}, v_i\}$ is an edge of X for $0 < i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $0 < i < s$
– or in other words, such that any two consecutive v_i are adjacent and any three consecutive v_i are distinct.

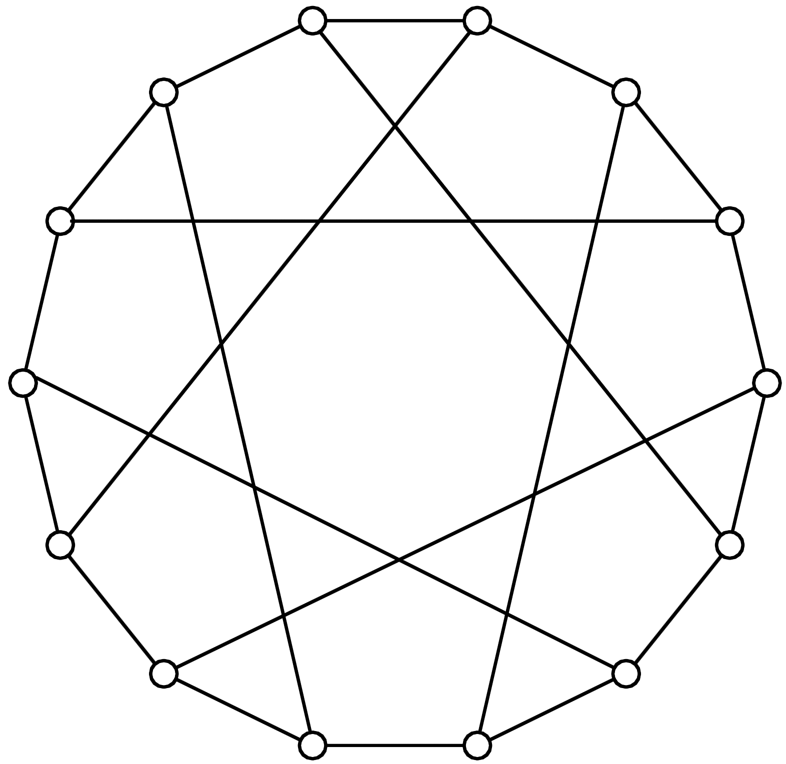
The graph X is called s-arc-transitive if its automorphism group $\text{Aut}(X)$ is transitive on the set of all s-arcs of X

The graph X is called s-arc-regular if $\text{Aut}(X)$ is regular (that is, sharply-transitive) on the set of all s-arcs of X — so given any two s-arcs $(v_0, v_1, v_2, \ldots, v_s)$ and $(w_0, w_1, w_2, \ldots, w_s)$ in X, there is unique automorphism of X taking v_i to w_i for all i.

Examples

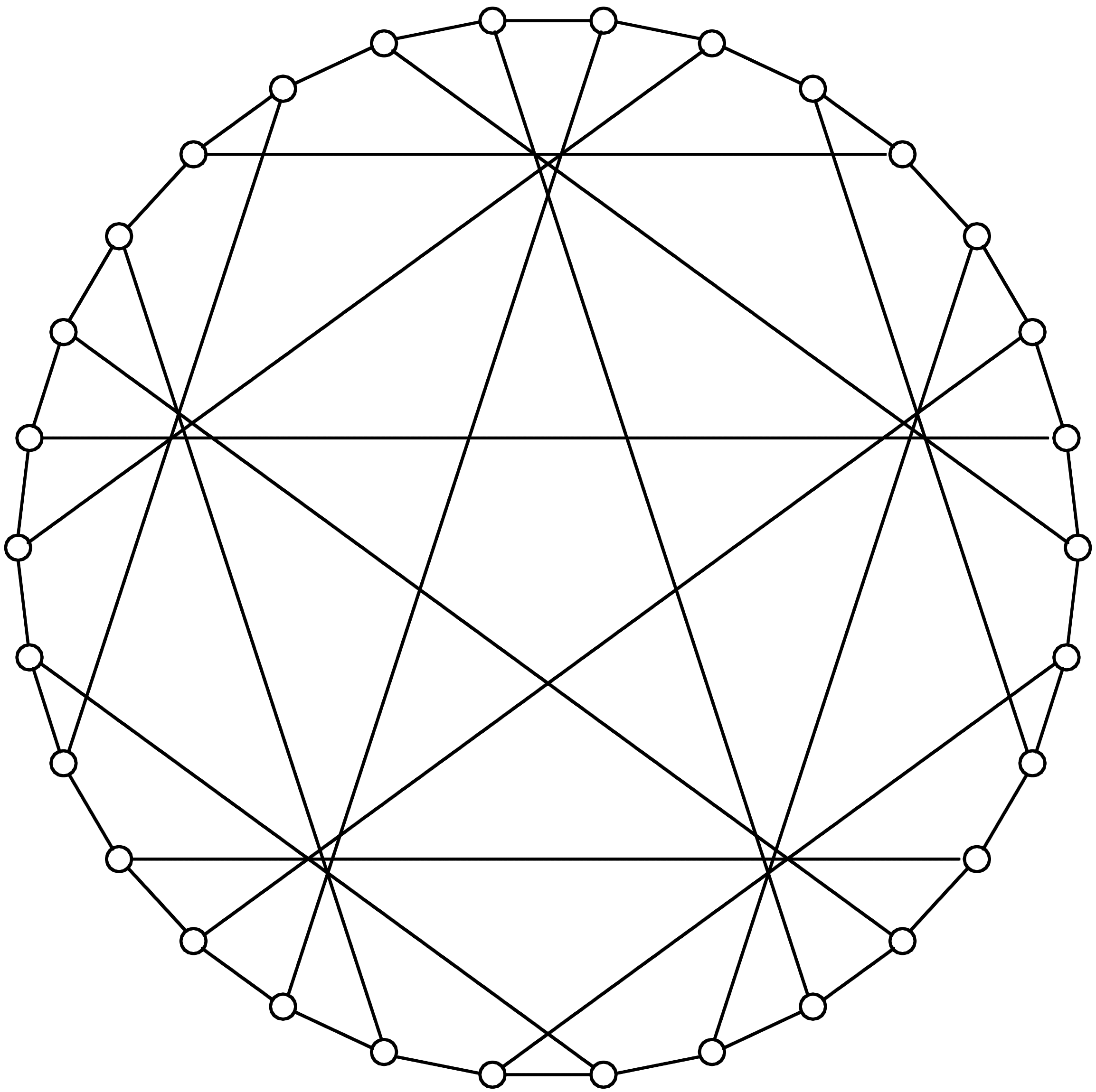
- C_n (n-cycle) is s-arc-transitive for all $s \geq 0$
- K_n is 2-arc-transitive but not 3-arc-transitive, for all $n > 3$
- $K_{n,n}$ is 3-arc-transitive but not 4-arc-transitive
- The cube graph Q_3 is 2-arc-regular (so not 3-arc-transitive)
- The Petersen graph is 3-arc-regular
- The Heawood graph (incidence graph of the Fano plane) is 4-arc-regular
- Tutte's 8-cage is 5-arc-regular.

Example: Heawood graph (4-arc-regular)



This is 4-arc-regular because the Fano plane is self-dual and its automorphism group is sharply 3-transitive on points

Tutte's 8-cage (This is 5-arc-regular)



Symmetric graphs: theorems of Tutte & Weiss

Let X be a finite connected symmetric (arc-transitive) graph.

- By vertex-transitivity, X is regular, so k-valent for some k

- If $k = 2$ then $X \cong C_n$ and $\mathrm{Aut}(X) \cong D_n$ for some n

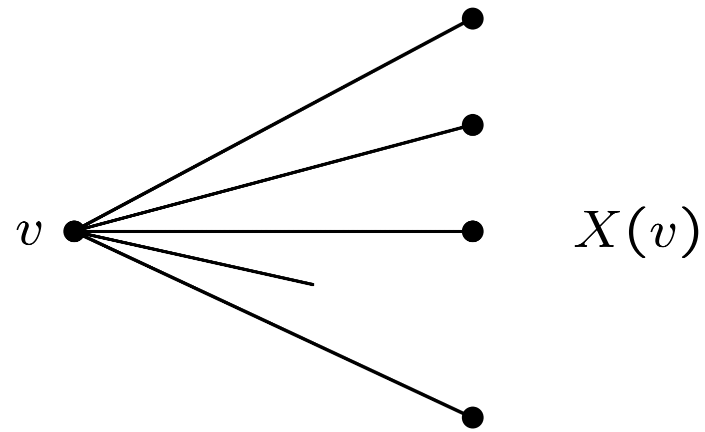
- Theorem (Tutte, 1947): If $k = 3$ then X is s-arc-regular and $|\mathrm{Aut}(X)| = 3 \cdot 2^{s-1} \cdot |V(X)|$ for some $s \in \{1, 2, 3, 4, 5\}$

[Proof takes about 3 pages, via combinatorial group theory]

- Theorem (Weiss, 1980): If $k > 3$ then X is s-arc-transitive for some $s \leq 7$, with $s = 7$ only when $k = 3^t + 1$ for some t

[Proof uses the classification of finite simple groups (CFSG)]

Local analysis



If X is s-arc-transitive for some $s \geq 2$, and $G = \mathrm{Aut}(X)$, then the stabiliser G_v of each vertex v is doubly-transitive on the set $X(v)$ of all neighbours of v.

Weiss's theorem uses the classification of doubly-transitive finite permutation groups (which depends on the CFSG) to determine and analyse possible actions of G_v on $X(v)$.

Cheryl Praeger and some of her colleagues are working on a (loose) classification of all 2-arc-transitive graphs.

Types of 3-valent symmetric graphs

By further local analysis, Djoković and Miller (1980) showed that if X is an s-arc-regular 3-valent graph, with $G = \mathrm{Aut}(X)$, then there are five possibilities for the vertex-stabiliser G_v, one for each $s \in \{1, 2, 3, 4, 5\}$. When $s = 2$ or 4 there are two different possibilities for the edge-stabiliser, but for $s = 1, 3$ and 5 there is just one each.

This gives seven types of finite 3-valent symmetric graphs, called 1, 2^1, 2^2, 3, 4^1, 4^2 and 5. Associated with each type is a universal group $\mathcal{U}$, such that the automorphism group of every example of that type is a quotient of $\mathcal{U}$.

The Foster census

In the 1930s, Ronald M. Foster (a mathematician/engineer working for Bell Labs) began a list of connected symmetric 3-valent graphs of order up to 512. This was published in 1988, and was remarkably good — with only a few gaps.

The census was extended (and the gaps filled) up to order 768 by MC and Peter Dobcsányi (2002), and recently by MC to order 10,000 (with the help of a new algorithm for finding finite quotients of finitely-presented groups). A list of all semi-symmetric 3-valent graphs of order up to 10,000 is also not far away.

Primož Potočnik, Pablo Spiga and Gabriel Verret have found all VT 3-valent graphs of order up to 1280 (this year), as well as all AT 4-valent graphs of order up to 640.

Pathological examples

Two of the classes of finite arc-transitive 3-valent graphs found by Djoković and Miller are interesting: 2^2 and 4^2. For the graphs in these classes, there are no edge-reversing automorphisms of order 2. But they could find no examples.

In the late 1980s we found some very large examples, of orders 6652800 and 29!/48 respectively, with automorphism groups S_{11} and A_{29}.

One of the gaps we filled in the Foster census was a type 2^2 graph of order 448, and this is the smallest such graph.

Open problem: What is the smallest arc-transitive 3-valent graph of type 4^2 (i.e. 4-arc-transitive but with no no edge-reversing automorphisms of order 2)?

Covering constructions

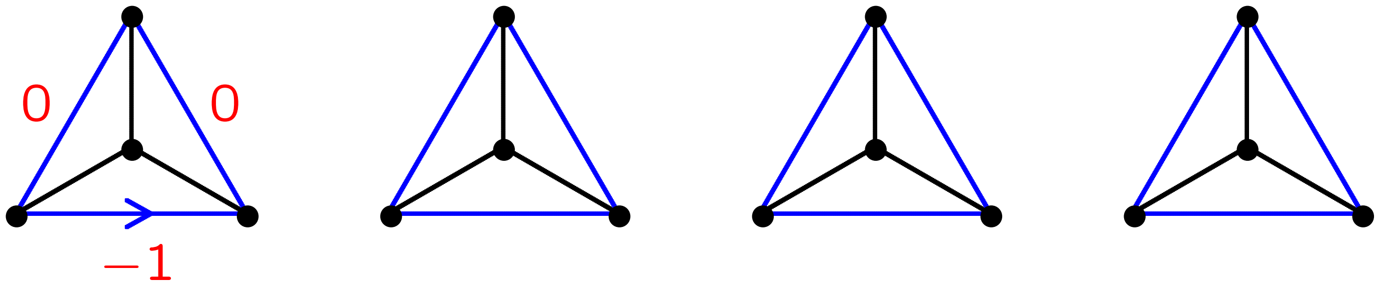
If $\theta\colon Y \to X$ is a locally bijective graph homomorphism, then X is a quotient of Y, and Y is a cover of X.

A (now) standard technique for constructing covers of graphs is to use voltage graphs:

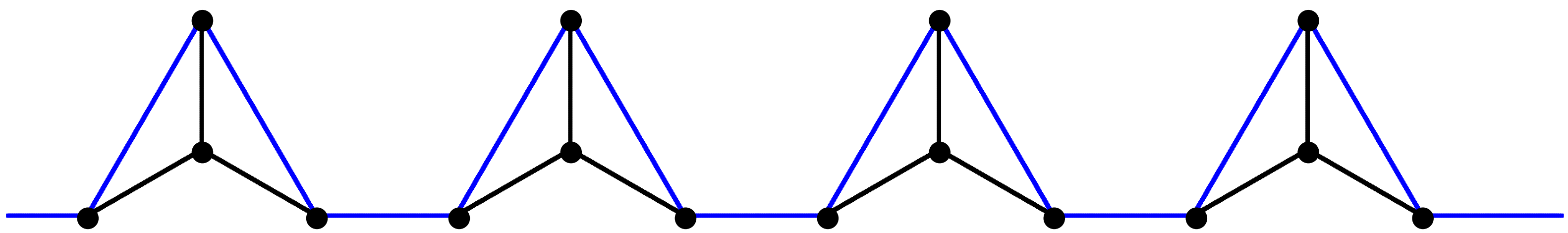
Let X be the base graph, and let A be a given group. Now assign elements of A to the arcs/darts of X, via some function $\nu\colon D(X) \to A$ (called a voltage assignment), in such a way that edges of a spanning tree T for X are assigned the identity element, and reversal of any arc inverts its voltage.

Define a covering graph $Y = Y_\nu$ with vertex-set $V(X) \times A$, and with (x, a) adjacent to (x', ab) whenever (x, x') is an arc of X with voltage assignment b.

Example: Covering K_4



Choose a spanning tree (black edges), assign 'voltages' to the free edges, each with a given orientation, and then use those free edges to link the copies together:



This gives a nice covering construction, that works well when A is cyclic or elementary abelian. The voltage group A can be viewed as being like a group of translations, permuting the copies of T.

An early version of this technique was used by John Conway in the 1970s to prove the existence of infinitely many connected 5-arc-transitive 3-valent graphs.

It was also used by Eyal Loz in his PhD thesis project (2005-2008) to construct graph covers with largest known order for given vertex-degree and diameter ... PTO1

And it has been applied to find all arc-transitive covers, or sometimes just 'semisymmetric' (edge-transitive but not vertex-transitive) covers of various small graphs, with cyclic or elementary abelian voltage groups ... PTO2

Degree-Diameter Table (as at December 2012)

$d \backslash k$	2	3	4	5	6	7	8	9	10
3	**10**	**20**	**38**	70	132	196	336	600	1 250
4	**15**	41	98	364	740	1 320	3 243	7 575	17 703
5	**24**	72	212	624	2 772	5 516	17 030	57 840	187 056
6	**32**	111	390	1 404	7 917	19 383	76 461	331 387	1 253 615
7	**50**	168	672	2 756	11 988	52 768	249 660	1 223 050	6 007 230
8	57	253	1 100	5 060	39 672	131 137	734 820	4 243 100	24 897 161
9	74	585	1 550	8 268	75 893	279 616	1 686 600	12 123 288	65 866 350
10	91	650	2 286	13 140	134 690	583 083	4 293 452	27 997 191	201 038 922
11	104	715	3 200	19 500	156 864	1 001 268	7 442 328	72 933 102	600 380 000
12	133	786	4 680	29 470	359 772	1 999 500	15 924 326	158 158 875	1 506 252 500
13	162	851	6 560	40 260	531 440	3 322 080	29 927 790	249 155 760	3 077 200 700
14	183	916	8 200	57 837	816 294	6 200 460	55 913 932	600 123 780	7 041 746 081
15	187	1 215	11 712	76 518	1 417 248	8 599 986	90 001 236	1 171 998 164	10 012 349 898
16	198	1 600	14 640	132 496	1 771 560	14 882 658	140 559 416	2 025 125 476	12 951 451 931
17	274	1 610	19 040	133 144	3 217 872	18 495 162	220 990 700	3 372 648 954	15 317 070 720
18	307	1 620	23 800	171 828	4 022 340	26 515 120	323 037 476	5 768 971 167	16 659 077 632
19	338	1 638	23 970	221 676	4 024 707	39 123 116	501 001 000	8 855 580 344	18 155 097 232
20	381	1 958	34 952	281 820	8 947 848	55 625 185	762 374 779	12 951 451 931	78 186 295 824

Some classifications of arc-transitive covers

- Cyclic coverings of Q_3 [Feng & Wang (2003)]
- Cyclic coverings of $K_{3,3}$ [Feng & Kwak (2004)]
- Elementary abelian covers of Petersen graph [Malnic & Potocnik (2006)]
- Semisymmetric elementary abelian covers of the Möbius-Kantor graph [Malnic, Marusic, Miklavic & Potocnik (2007)]
- Elementary abelian covers of Pappus graph [Oh (2009)]
- Elementary abelian covers of the octahedral graph [Kwak & Oh (2009)]
- Elementary abelian covers of K_5 [Kuzman (2010)]

A new method for finding covers

We now have a new method for finding all arc-transitive regular covers of a given graph with abelian covering group, using a group-theoretic approach that is more helpful than the 'voltage-graph' technique.

In joint work with PhD student Jicheng Ma (2009–2012) we now have all such covers of K_4, $K_{3,3}$, Q_3, the Petersen graph and the Heawood graph. Jicheng is now working on finding all arc-transitive abelian regular covers of K_5.

The same ideas can be used to find regular covers of maps and polytopes (which are like 'higher-dimensional' maps).

Graphs with A_n or S_n as automorphism group

By Tutte's theorem, the 'most symmetric' of the finite arc-transitive 3-valent graphs are 5-arc-transitive.

Small examples include Tutte's 8-cage (order 30), a triple-cover of Tutte's 8-cage (order 90), Wong's graph (order 234) and a double-cover (order 468), the Sextet graph $S(5)$ (order 650), and the Biggs-Conway graph (order 2352).

One might consider these graphs to be rare, but in some sense they are not: for every sufficiently large n, there exists at least one 5-arc-transitive 3-valent graph with automorphism group A_n, and at least one 5-arc-transitive 3-valent graph with automorphism group S_n.

Graphs with A_n or S_n as autom group (cont.)

The same thing holds for all seven of the Djoković-Miller types of arc-transitive 3-valent graphs, and also for 7-arc-transitive 4-valent graphs (the best possible for valency 4).

In each case, the 'universal' group $\mathcal{U}$ for the class of arc-transitive actions is an amalgamated free product $V *_A E$, where V, A and E are the stabilisers of a vertex v, an arc (v, w) and the edge $\{v, w\}$ respectively.

Conjecture (open problem): If V and E are finite groups with a common subgroup A such that A has index at least 3 in V and at least 2 in E, then all but finitely many alternating groups A_n are quotients of $\mathcal{U} = V *_A E$.

Locally arc-transitive graphs

A graph X is locally s-arc-transitive if the stabiliser in $\mathrm{Aut}(X)$ of a vertex v is transitive on all s-arcs in X starting at v.

Theorem (Stellmacher (1996), unpublished): If X is a finite locally s-arc-transitive graph, then $s \leq 9$.

Until recently, the only known examples for $s = 9$ came from classical generalised octagons and their covers. Such graphs are edge-transitive and bipartite, but not vertex-transitive.

Indeed they need not even be regular: vertices in different parts can have different valencies.

Locally arc-transitive graphs (cont.)

The smallest example for $s = 9$ has order 4680, with vertices of valency 3 in one part and 5 in the other. Its automorphism group is ${}^2F_4(2)$ (a Ree simple group), with vertex-stabilisers U and V of orders 12288 and 20480, and arc/edge-stabiliser $A = U \cap V$ of order 4096.

Theorem [MC (2011), not yet written up]

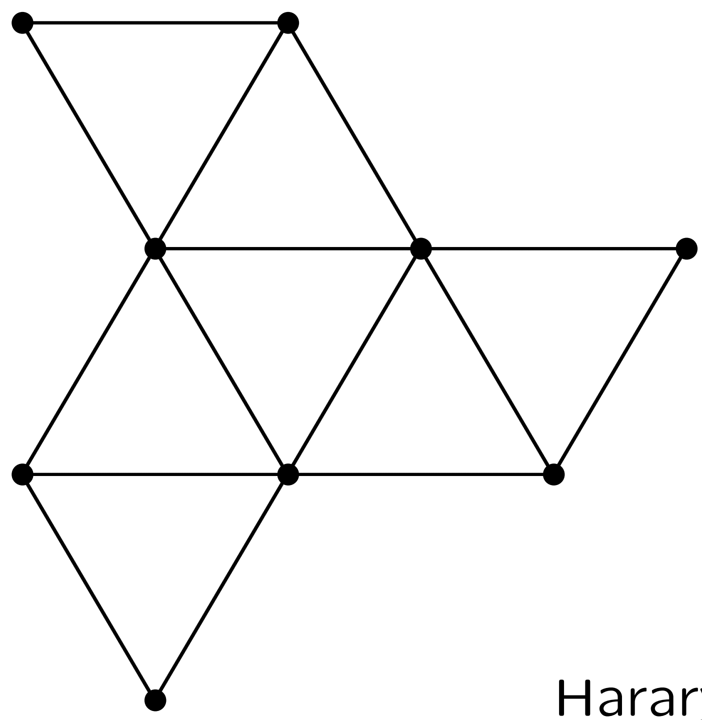
The free product $U *_A V$ has all but finitely many A_n as quotients. Hence for all but finitely many n, the alternating group A_n is the automorphism group of an edge-transitive, locally 9-arc-transitive bipartite graph (with vertices of valency 3 in one part and 5 in the other).

THANK YOU

THANK YOU

Answers to questions

- The smallest graph with exactly 3 symmetries?

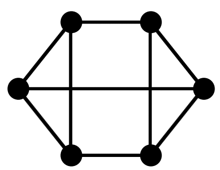


Harary & Palmer (1960s)

- The connected 3-valent graphs of order 4, 6, 8 and 10 with the fewest symmetries?

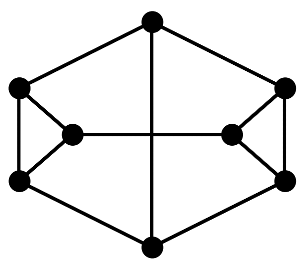
Order 4: K_4 (the only such graph of order 4)

Order 6:



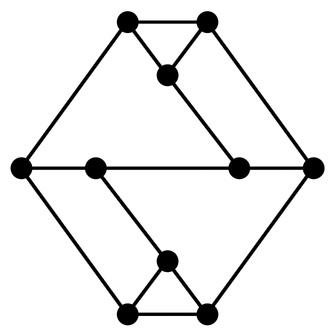
12 symms

Order 8:



4 symms

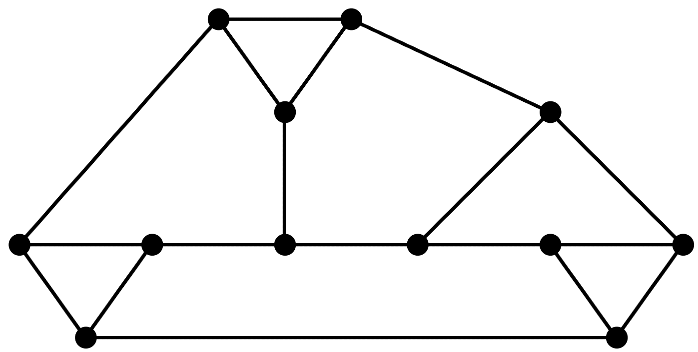
Order 10:



2 symms

- The smallest 3-valent graph with just one symmetry?

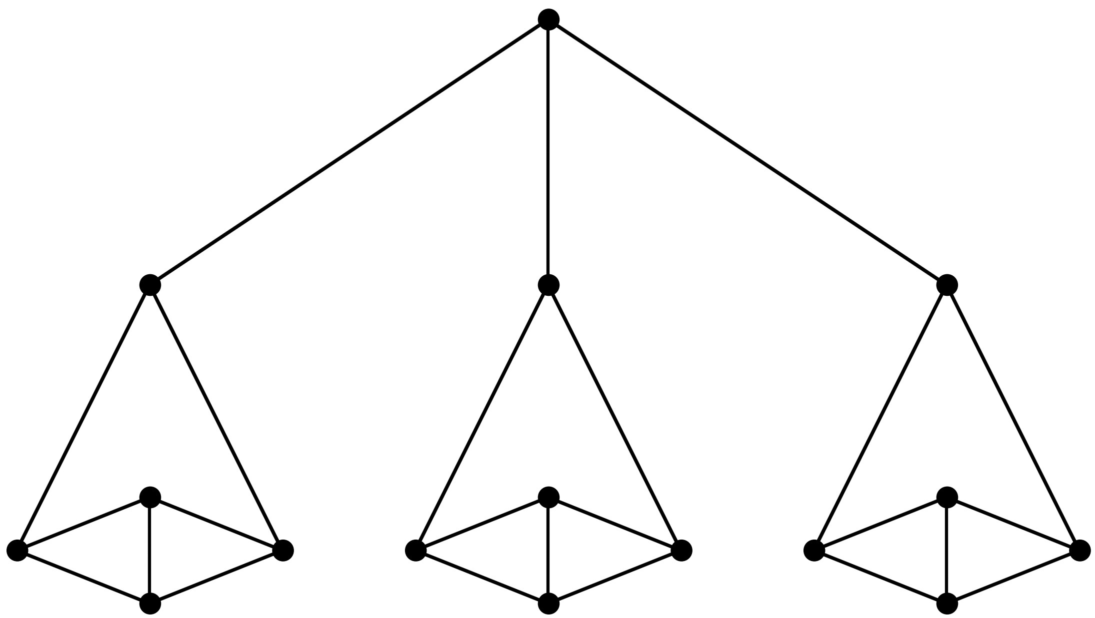
  Order 12:



  This is the Frucht graph, found by Robert Frucht in 1939.

- The connected 3-valent graph of order 16
  with the most symmetries?

NOT the Moebius-Kantor graph (with 96 symmetries), but this graph:



... which has $3! \cdot (2 \cdot 2)^3 = 384$ symmetries